Mathematical Theory of Rational Behavior

and Potential Applications to Resilient Monitoring/Control

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MOTIVATION

- Bees caste regulation process:
  - M bees in S castes
  - $\nu(i)$ is fraction of bees in caste i, $i = 1, \ldots, S$
  - Remove on the castes (experimentally, foragers)
  - In a short time, bees re-distribute themselves among the castes so that $\nu(i)$'s remain the same
- Questions:
  - How do the bees monitor the “plant” (family)?
  - How do they control the plant (determine the optimal distribution of bees among castes and maintain it)?

- First goal: Develop a theory that could explain (at least, hypothetically) this phenomenon
- Second goal: Apply this theory to resilient monitoring and control of industrial plants
- The first goal has been, to a certain degree, accomplished in S.M. Meerkov, "Mathematical Theory of Rational Behavior", *Mathematical Biosciences*, 1979
- The second is being pursued today in a recently initiated resilient monitoring project with INL (Dr. Garcia)
- The purpose of this talk to overview TRB and illustrate it by an application in a traffic control problem

OUTLINE

1. Individual rational behavior
2. Group rational behavior
3. Application
4. Potentials in resilient monitoring/control
5. Open problems
6. Conclusions

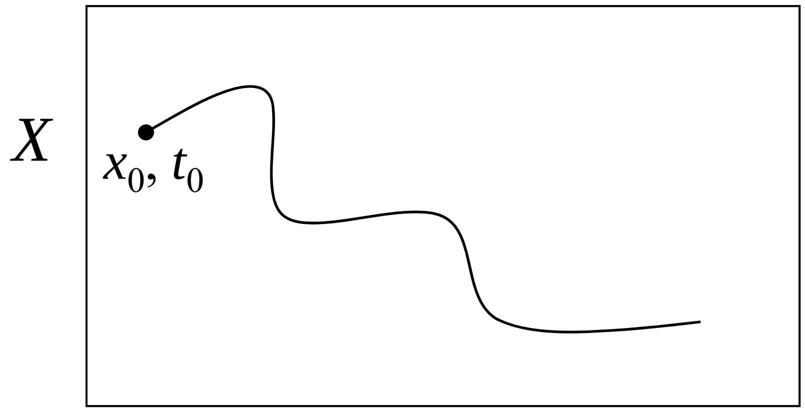
1. MODELING AND ANALYSIS OF INDIVIDUAL RATIONAL BEHAVIOR

1.1 Rational Behavior

- *Behavior* – a sequence of decisions in time, i.e., a dynamical system in the decision space X:

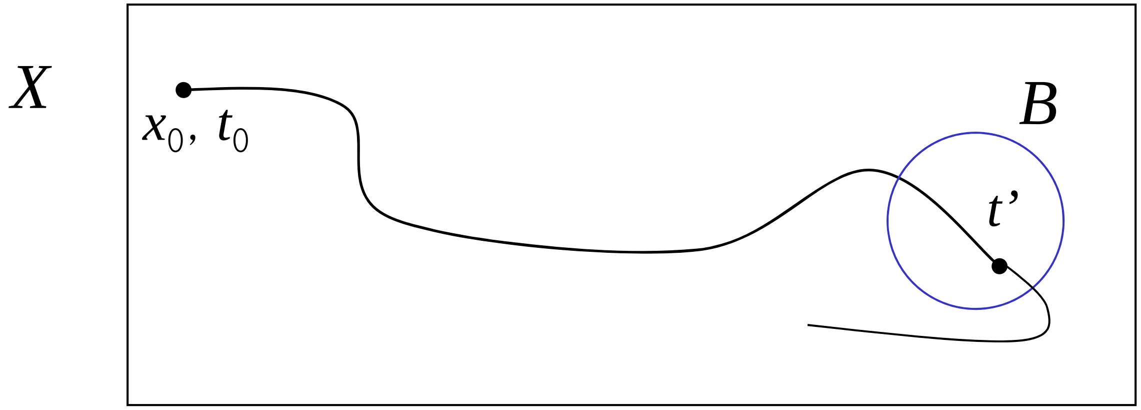
$$x_{\varphi(x),N}(x_0, t_0, t),$$

$$\varphi(x) > 0, \forall x \in X, N \in \{1, 2, \ldots\}.$$



- *Rational behavior* – the behavior $x_{\varphi(x),N}(x_0, t_0, t)$, which satisfies the following axioms:
  - *Ergodicity*:

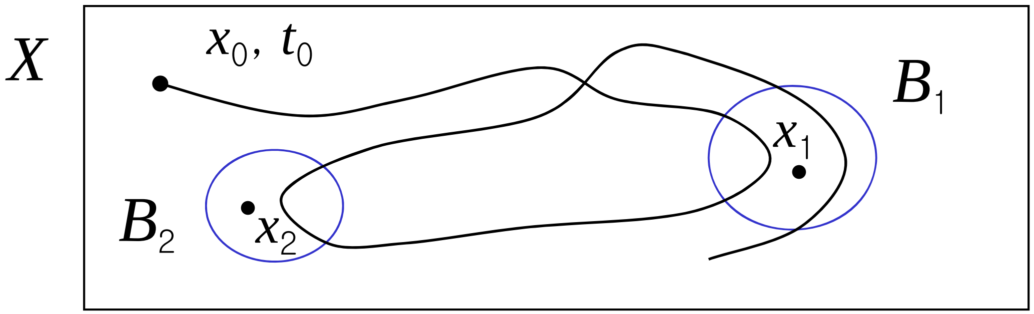
$$\forall x_0, t_0, \forall B \subset X, \mu B > 0, \exists t' \text{ such that}$$
$$x_{\varphi(x),N}(x_0, t_0, t') \in B$$



- *Selectivity* or *rationality*:

$$\forall x_0, t_0, \forall x_1, x_2 \in X \text{ and } B_1, B_2 \subset X, \mu B_i > 0, B_1 \cap B_2 = \phi$$

$$\frac{T_{B_1}}{T_{B_2}} > 1 \text{ if } \varphi(x_1) < \varphi(x_2).$$



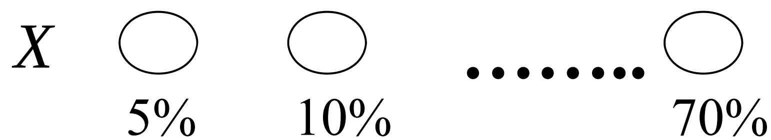
$$\text{Moreover, } \quad \frac{T_{B_1}}{T_{B_2}} \to \infty \text{ as } N \to \infty.$$

- $\varphi(x) \to$ penalty function at decision x
- $N \to$ measure of rationality

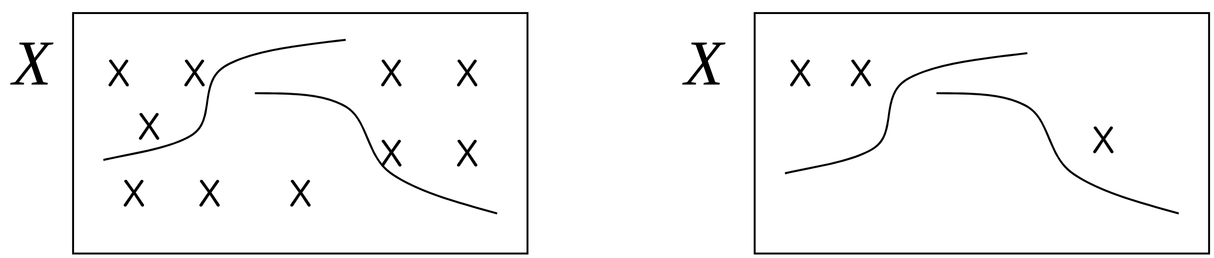
1.2 Examples of Rational Behavior

1.2.1 Natural systems

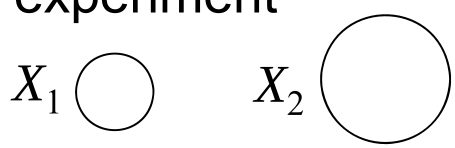
- Bees in foraging behavior



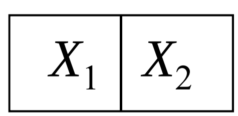
- Mice in feeding behavior



- Dog in the circle experiment



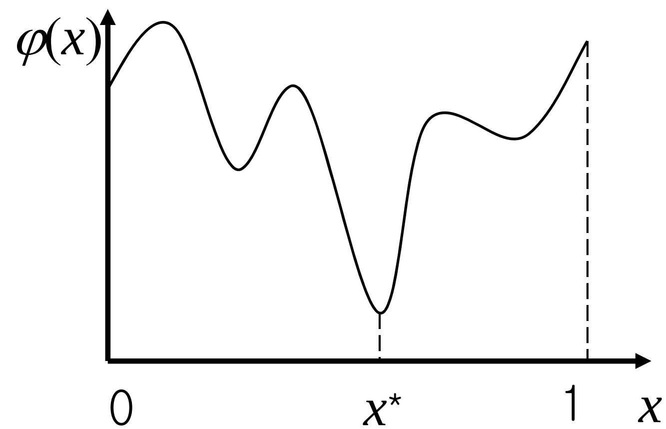
- Workers in production (Safelite Glass, Lincoln Electric)



1.2.2 Mathematical systems

- Ring element: $X = [0,1)$

$$\dot{x} = \varphi^N(\{x\}), \quad x(0) = 0.$$



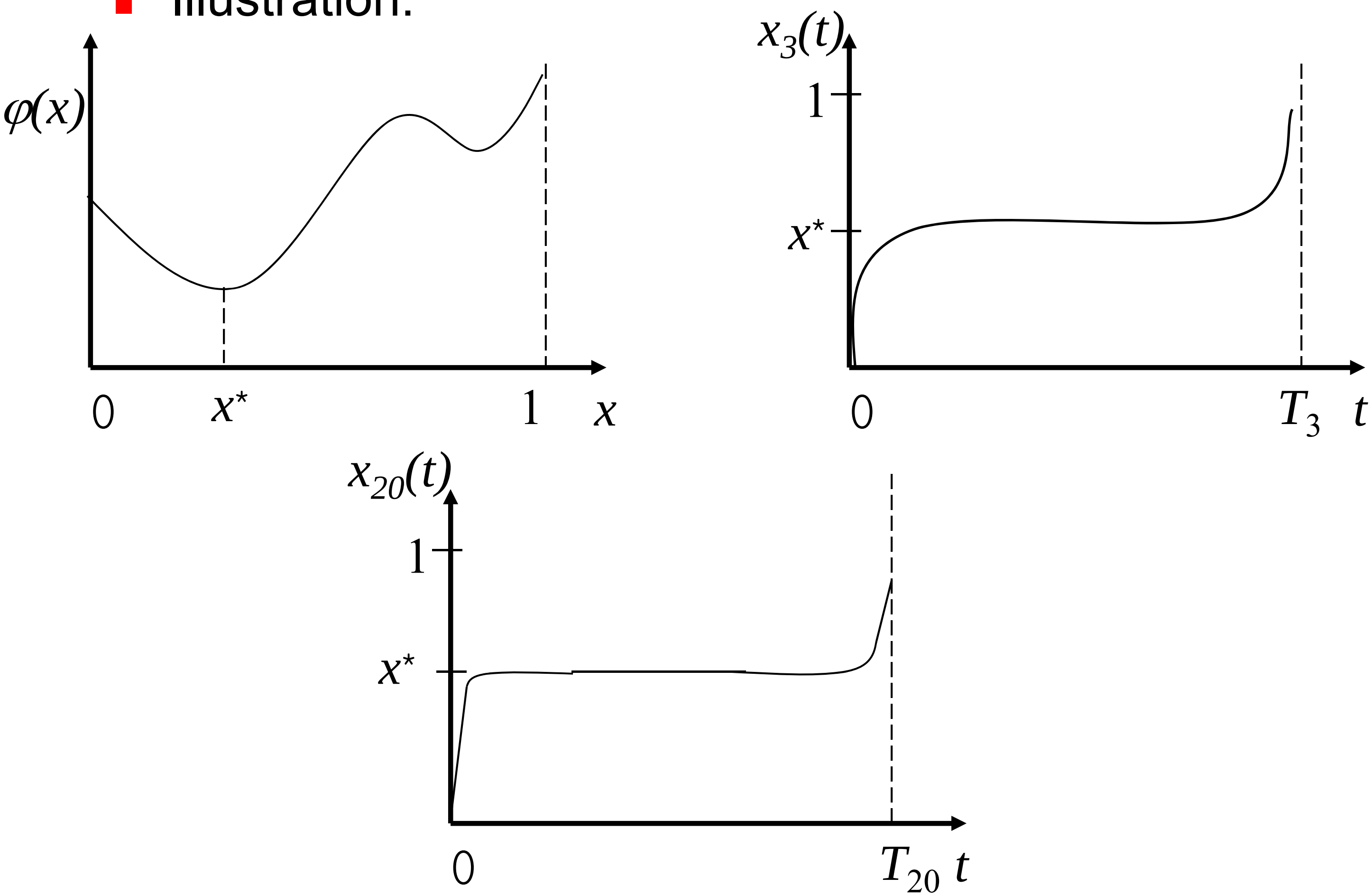
  - Ergodicity takes place
  - Rationality:

$$\frac{T(x_1)}{T(x_2)} \approx \alpha \left[\frac{\varphi(x_2)}{\varphi(x_1)} \right]^N$$

  - Additional property:

$$\lim_{N\to\infty} \frac{1}{T_N} \int_0^{T_N} x_N(t) = x^*, \quad x^* = \arg \inf_{x\in[0,1]} \varphi(x).$$

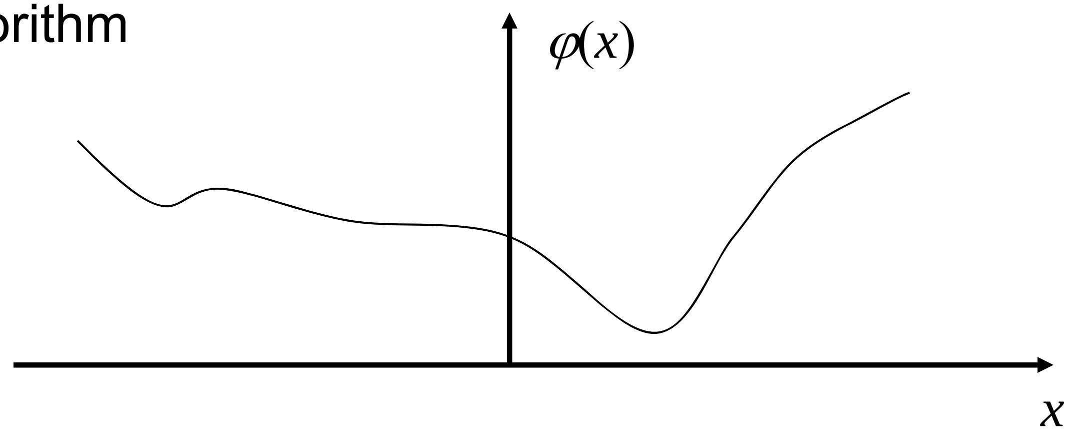
- Illustration:



- A search algorithm

$$X = \mathbf{R}$$

$$\varphi(x) > 1, \forall x \in \mathbf{R}$$



$$dx = -\frac{\partial \varphi^N}{\partial x} dt + dw$$

$$p(x) = Ce^{-\varphi^N(x)}, \ \forall x \in \mathbf{R}$$

$$\frac{p(x_1)}{p(x_2)} = e^{\varphi^N(x_2) - \varphi^N(x_1)} \xrightarrow{N \to \infty} \infty$$

2. MODELING AND ANALYSIS OF GROUP RATIONAL BEHAVIOR

2.1 Groups of Rational Individuals

- *Group* – a set of $M > 1$ rational individuals interacting through their penalty functions:

$$\varphi_i = \varphi_i(x_1, \dots, x_i, \dots, x_M), \quad i = 1, \dots, M$$

- Group state space:

$$x \in X = X_1 \times X_2 \times \cdots \times X_M$$

- Sequential algorithm of interaction:

$$\varphi_i = \varphi_i(x_1 = \text{const}, \dots, x_i = \text{var}, \dots, x_M = \text{const}), \quad i = 1, \dots, M$$

2.2 Homogeneous Fractional Interaction

- M individuals, $X_i = X, \ \forall i$

X

X_1	X_2

- Assume that at t_0:

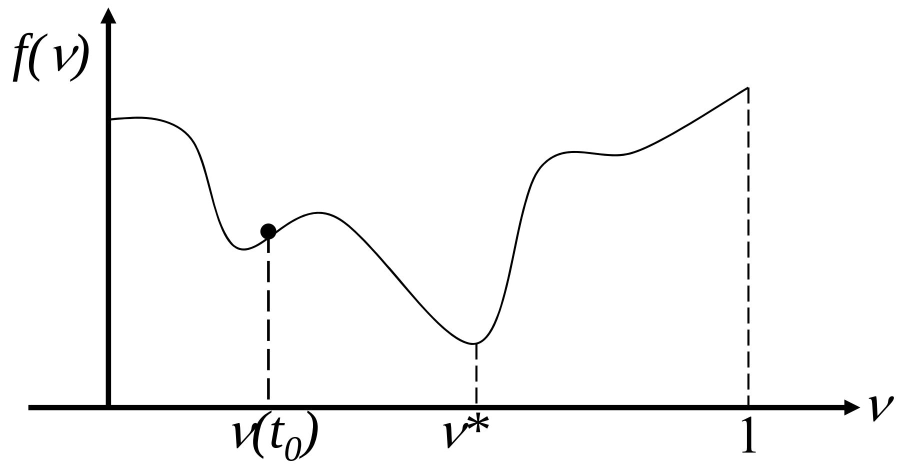
$$m(t_0) \text{ in } X_1,$$

$$M - m(t_0) \text{ in } X_2,$$

$$\nu(t_0) = \frac{m(t_0)}{M}.$$

- *Homogeneous Fractional Interaction* – an interaction defined by the group penalty function:

$$f(\nu) > 0, \quad \nu \in [0, 1]$$

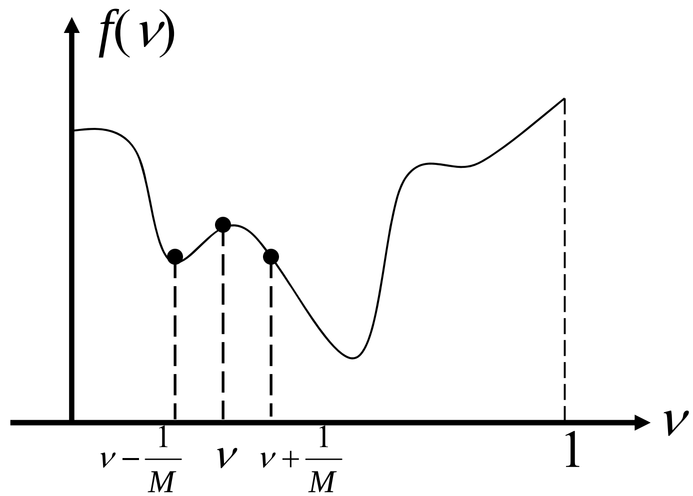


- Group penalty function defines the penalty function of each individual as follows:
  - For $x_i(t_0) \in X_1$,

$$\varphi_i = \begin{cases} f(\nu), & \text{if } x_i \in X_1, \\ f(\nu - \frac{1}{M}), & \text{if } x_i \in X_2. \end{cases}$$

  - For $x_i(t_0) \in X_2$,

$$\varphi_i = \begin{cases} f(\nu + \frac{1}{M}), & \text{if } x_i \in X_1, \\ f(\nu), & \text{if } x_i \in X_2. \end{cases}$$



- Interpretation
  - Beehive food distribution
  - Corporation-wide bonuses
  - Uniform wealth distribution
- Desirable state

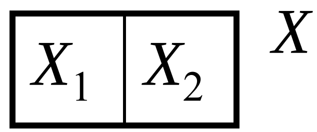
$$v^* = \arg \inf_{v \in [0,1]} f(v)$$

- Question:

$$v(t) \to v^* ?$$

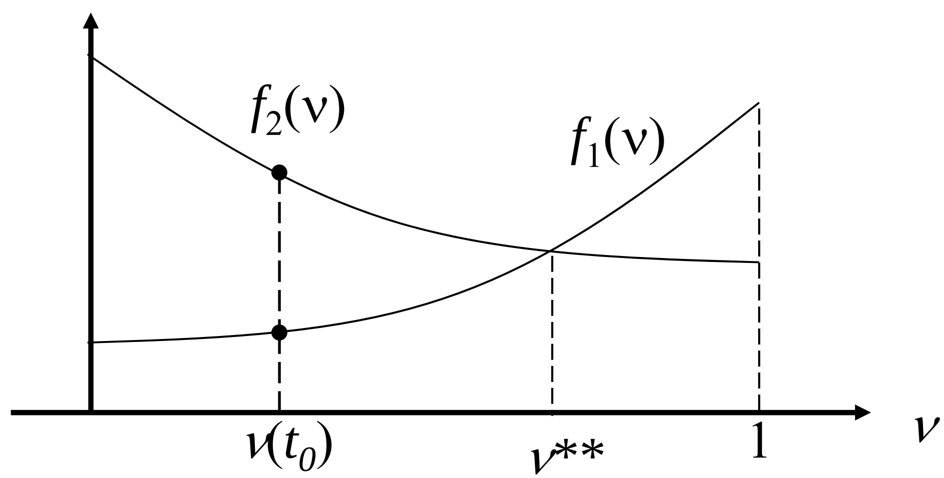
2.3 Inhomogeneous Fractional Interaction

- M individuals, $X_i = X, \ \forall i$



- *Inhomogeneous Fractional Interaction* – an interaction defined by two subgroup penalty functions

$$f_1(v) > 0, \ \ f_2(v) > 0, \ \ v \in [0,1]$$

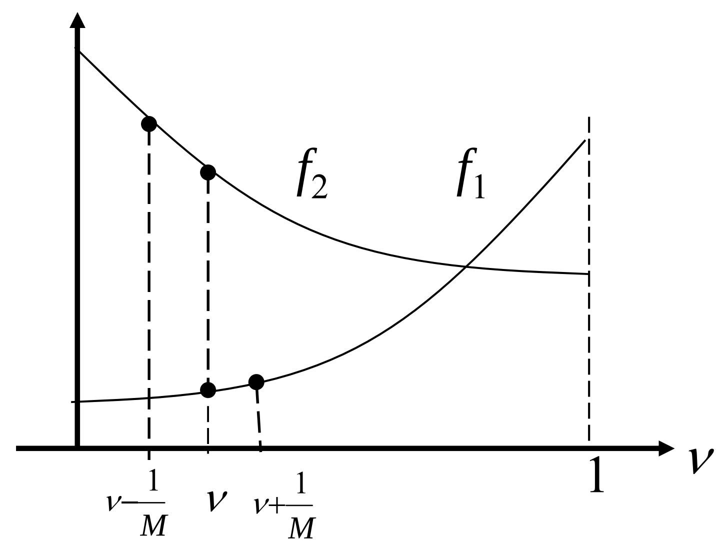


- Penalty for each individual are defined as follows:
  - For $x_i(t_0) \in X_1$,

$$\varphi_i = \begin{cases} f_1(\nu), & \text{if } x_i \in X_1, \\ f_2(\nu - \dfrac{1}{M}), & \text{if } x_i \in X_2. \end{cases}$$

  - For $x_i(t_0) \in X_2$,

$$\varphi_i = \begin{cases} f_1(\nu + \dfrac{1}{M}) & \text{if } x_i \in X_1, \\ f_2(\nu) & \text{if } x_i \in X_2. \end{cases}$$



- Interpretation
  - Differentiated corporate bonuses system
  - Non-uniform wealth distribution

- Desirable state: Nash equilibrium

$$\nu^{**} = \arg[\, f_1(\nu) = f_2(\nu)]$$

- Question:

$$v(t) \to v^{**}\,?$$

2.4 Properties of Group Behavior under Homogeneous Fractional Interaction

- **Theorem:** Under the homogeneous fractional interaction, the following effect of "critical mass" takes place: $\exists\ C > 0$, such that

$$\nu(t) \xrightarrow{p} \nu^* \text{ if } \lim_{N,M \to \infty} \frac{N}{M} \geq C$$

$$\nu(t) \xrightarrow{p} 0.5 \text{ if } \lim_{N,M \to \infty} \frac{N}{M} = 0.$$

- $\nu = 0.5$ implies the state of maximum entropy – the group behaves like a statistical mechanical gas (no rationality)

- Empirical observations

  - Beehive: when *M is* sufficiently small, the caste regulation process takes place; when *M* becomes large, the family splits
  - Abnormal behavior of unusually large groups of animals (locust, deers, etc.)
  - Pay-for-group-performance: cooperate-wide bonuses, BP – Prudhoe Bay vs. Anchorage

2.5 Properties of Group Behavior under Inhomogeneous Fractional Interactions

- Assume $\exists$ unique v^{**} such that $f_1(v^{**}) = f_2(v^{**})$ and

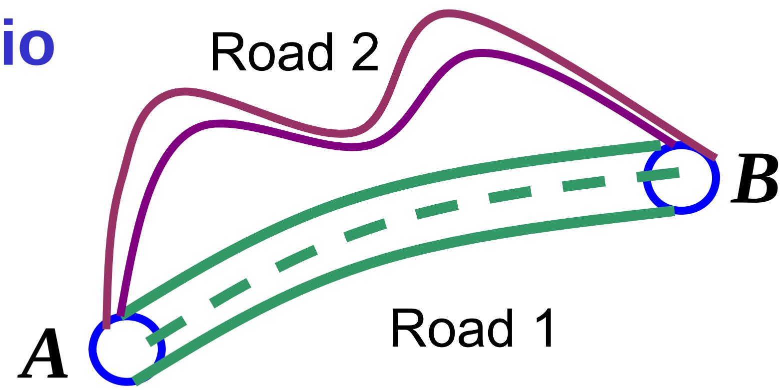
$$f_2(v^{**} - \Delta) > f_1(v^{**} - \Delta),$$
$$f_2(v^{**} + \Delta) < f_1(v^{**} + \Delta),$$
$$0 < \Delta << 1.$$

- **Theorem:** Under the inhomogeneous fractional interaction, no effect of "critical mass" takes place. Specifically, $\exists\, N^*$ such that $\forall N \geq N^*$

$$v(t) \xrightarrow{p} v^{**} \text{ for } \forall\, M.$$

3. APPLICATION TO PAY AND INCENTIVE SYSTEM (Joint work with UM undergraduate Leeann Fu)

3.1 Scenario



Penalty function: $\text{travel time } t_i = t_{0_i}\left(1 + K\frac{x_i}{1-x_i}\right), \quad i = 1, 2,$

$x_i = \dfrac{n_i}{c_i}$ – the level of road congestion,

c_i – road capacity,

n_i – number of vehicles on the road,

$K \in (0,1]$ – road condition factor.

- *Problem* 1: Assuming that the performance index is time to travel and each driver exhibits rational behavior, investigate the steady state distribution of vehicles among Roads 1 and 2
- *Problem* 2: Assuming that the drivers are rational and given a fixed amount of goods to transport from *A* to *B*, analyze the total time necessary to transport the goods under different pay systems:
  - Pay-for-individual-performance
  - Pay-for-group-performance
  - Pay-for-time

3.2 Parameters Selected

- $M = 6$
- $N = \text{var}$
- Road systems
  - System 1:

$$t_{0_1} = 2.05,\ K = 0.7,\ c = 9$$

$$t_{0_2} = 2.4,\ K = 1.0,\ c = 8$$

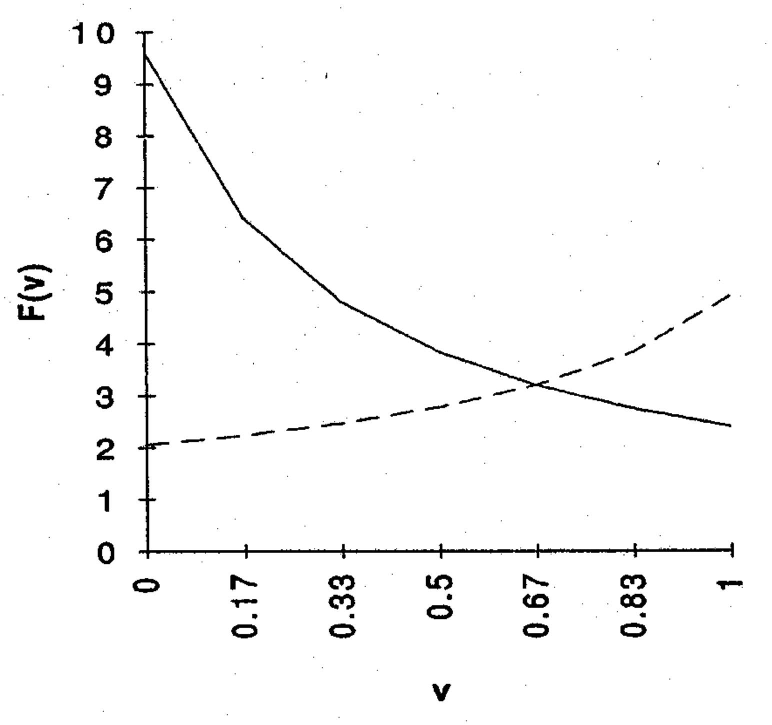
  - System 2:

$$t_{0_1} = 2.1,\ K = 0.68,\ c = 10$$

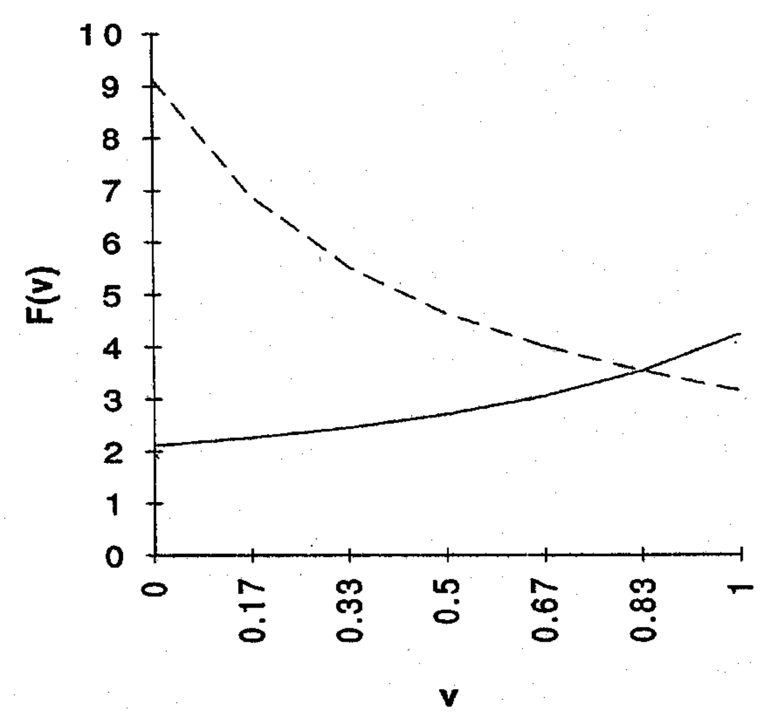
$$t_{0_2} = 3,\ K = 0.95,\ c = 9$$

3.3 Problem 1

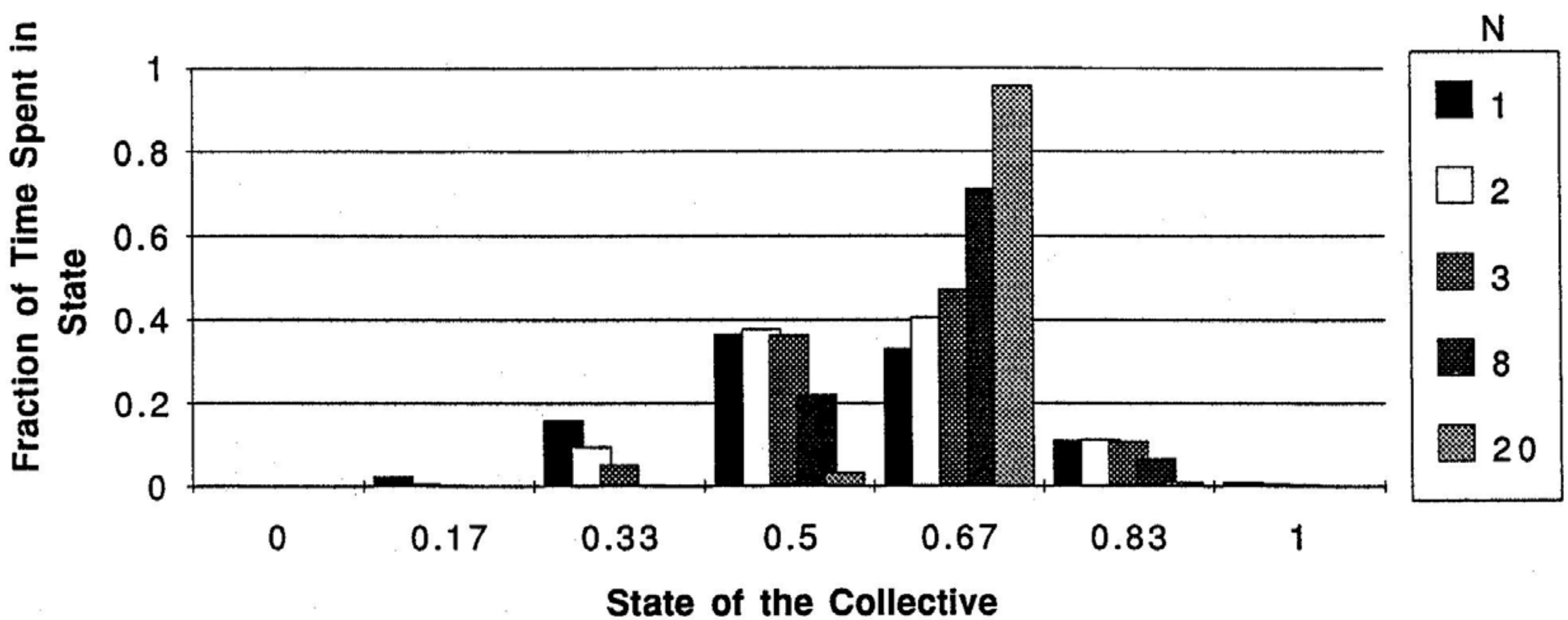
- Penalty functions
  - System 1:



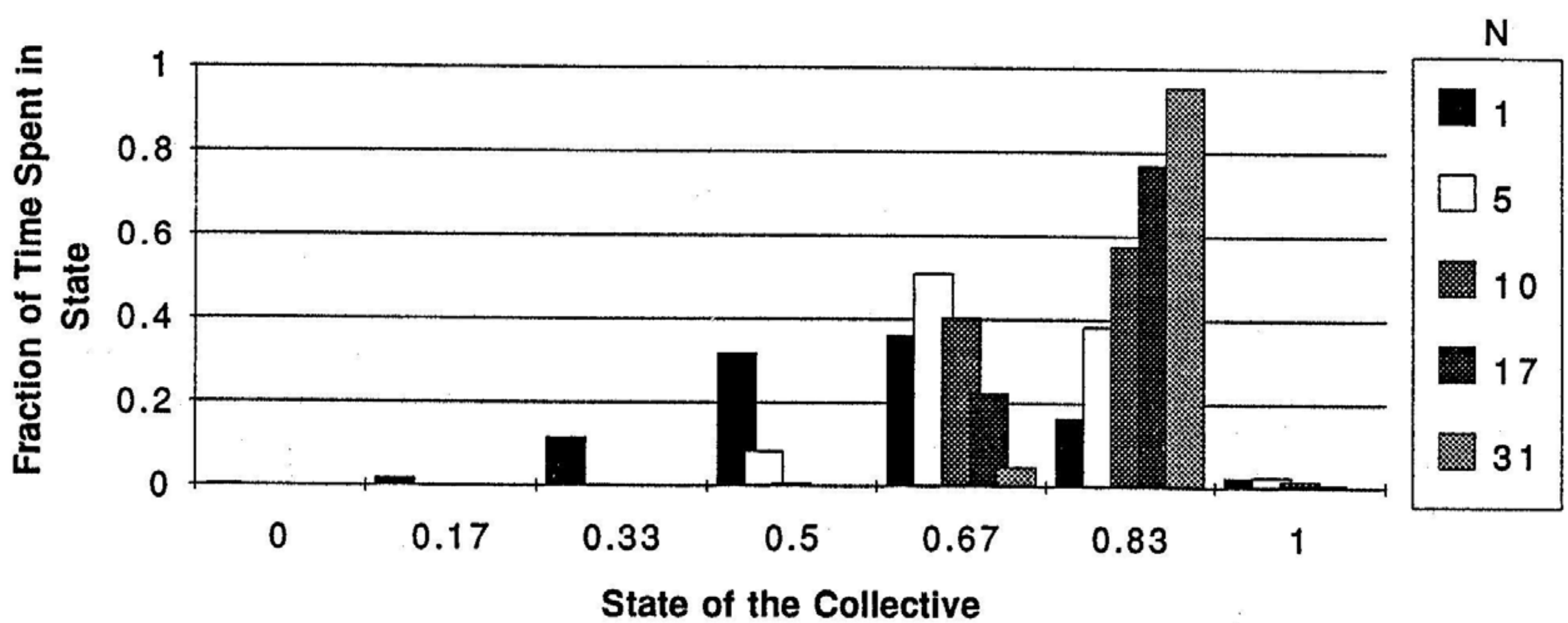
  - System 2:



- Results
  - System 1:



- System 2:



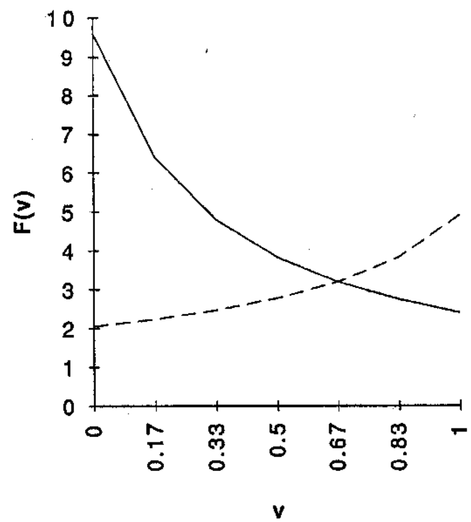
Fraction of Time Spent in State
1
0.8
0.6
0.4
0.2
0
0
0.17
0.33
0.5
0.67
0.83
1
State of the Collective
N
1
5
10
17
31

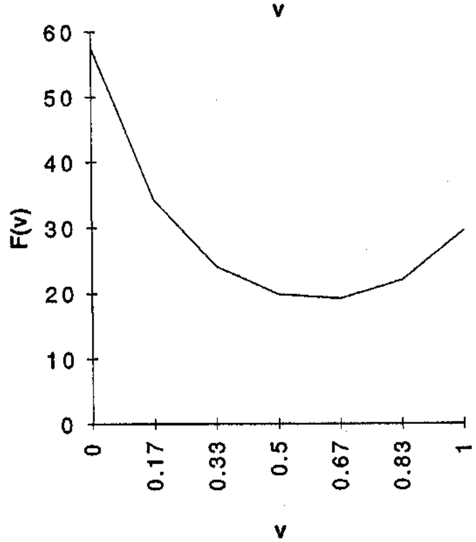
3.4 Problem 2

- Penalty functions
  - System 1:

User eq. = System eq.:

$$v^* = v^{**}$$

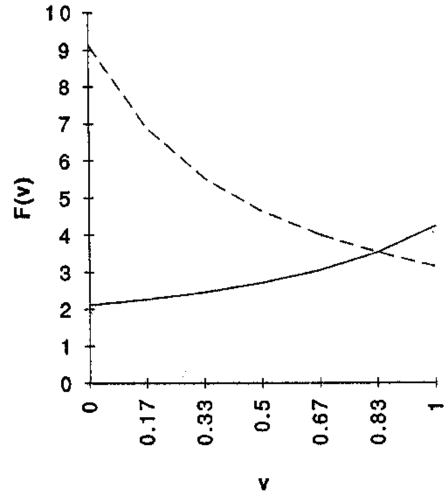


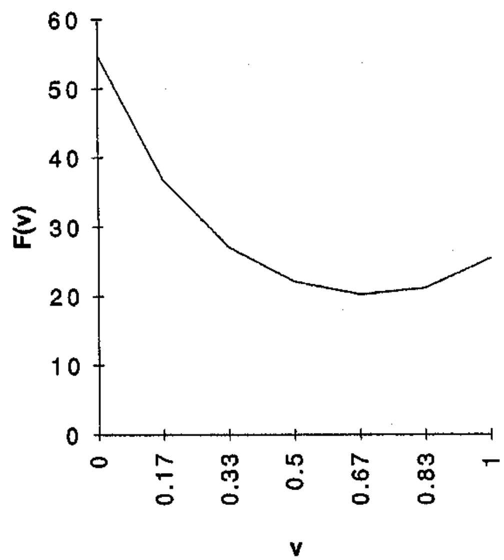


- System 2:

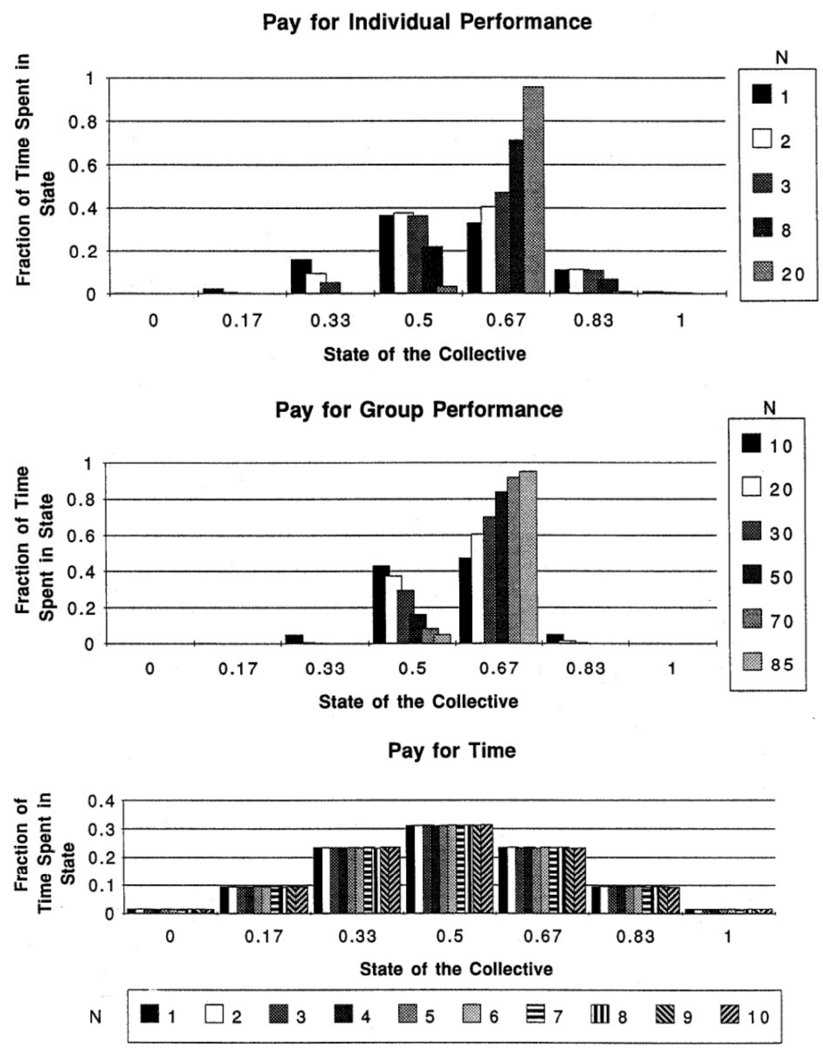
User eq. ≠ System eq.:

$$v^* \neq v^{**}$$

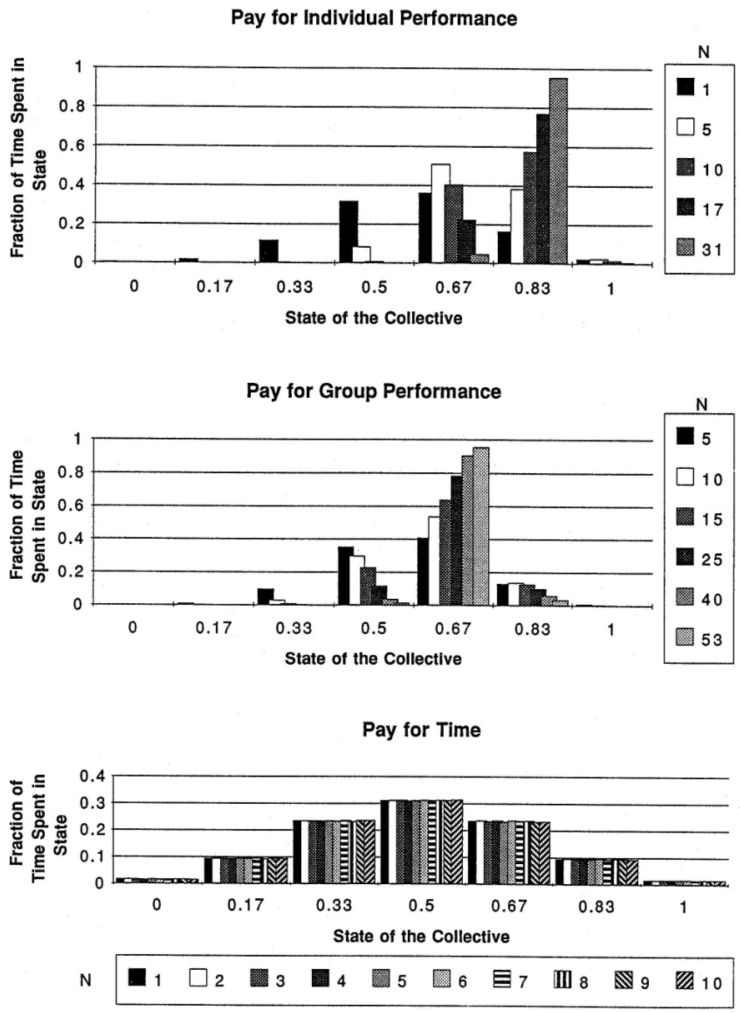




- Results
  - System 1



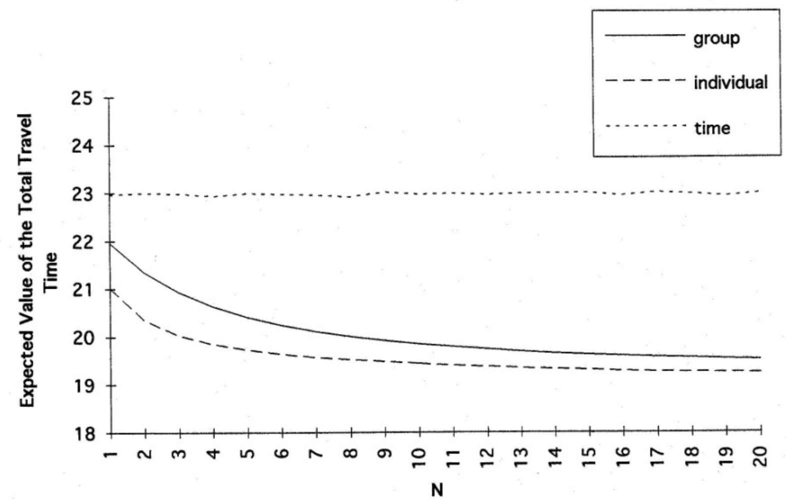
- System 2



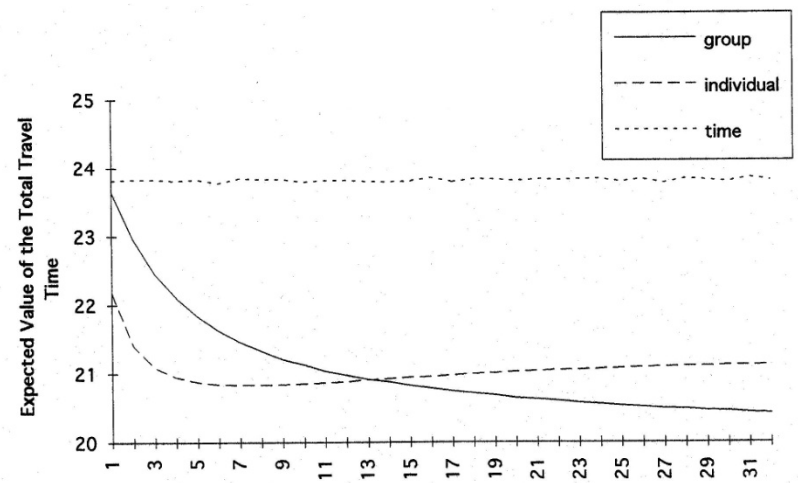
Pay for Individual Performance
Fraction of Time Spent in State
1
0.8
0.6
0.4
0.2
0
0
0.17
0.33
0.5
0.67
0.83
1
State of the Collective
N
1
5
10
17
31
Pay for Group Performance
Fraction of Time Spent in State
1
0.8
0.6
0.4
0.2
0
0
0.17
0.33
0.5
0.67
0.83
1
State of the Collective
N
5
10
15
25
40
53
Pay for Time
Fraction of Time Spent in State
0.4
0.3
0.2
0.1
0
0
0.17
0.33
0.5
0.67
0.83
1
State of the Collective
N
1
2
3
4
5
6
7
8
9
10

3.5 Comparisons

- For system 1



- For system 2



3.6 Discussion

- User equilibrium = system equilibrium ($v^* = v^{**}$): pay-for-individual-performance is the best

- User equilibrium ≠ system equilibrium ($v^* \neq v^{**}$): pay-for-group-performance maybe the best (if M is sufficiently small and N is sufficiently large)

4. POTENTIAL APPLICATIONS IN RESILIENT MONITORING/CONTROL

- Resilient Monitoring:
  - sensor allocation
  - sensor regime optimization
  - sensor spatial distribution
- Resilient Control:
  - control laws for rational controllers
  - analysis of closed loop systems with rational controllers
  - actuator and sensor re-allocation
  - non-standard control problems (e.g., robot colonies)

5. OPEN PROBLEMS

- Learning in the framework of rational behavior
  - Modeling of experience-based learning
  - Analysis of rational behavior with learning
- Groups of individuals with different levels of rationality
- Group behavior under rules of interaction other than fractional
- General theory of rational deciders

5. CONCLUSION

- Mimicking physical potentials of natural systems led to airplanes, car, computers, radars, etc.

- Mimicking the capacity of natural systems to resiliency and adaptation will lead to mechanisms that can survive in the artificial world of the "survival of the fittest".