# Musings on Persistent Excitation Prompts New Weighted Least Squares SysID Method for Nonlinear Differential Equation Based Systems

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# Outline


1. The Problem
2. Examples and Persistent Excitation
3. Conclusions and Future Directions

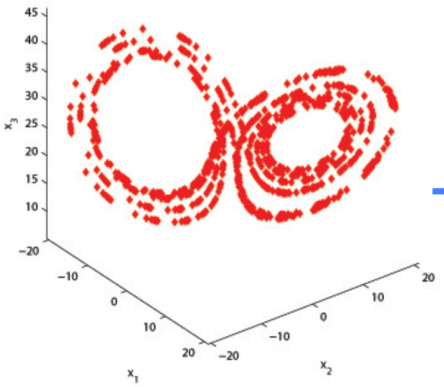
# Problem motivation

This research started as part of INL's resilient control system research program

- The first step in any control or prognostication system development is the creation of a system model
- There exist three basic types of models:

**White Box** $\leftrightarrow$ **Gray Box** $\leftrightarrow$ **Black Box**

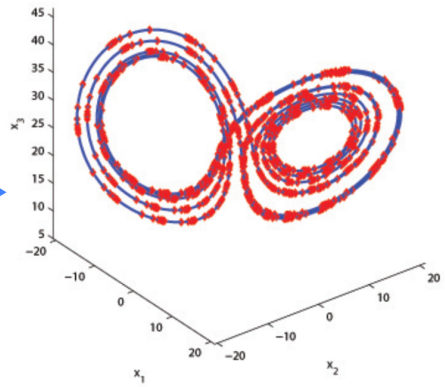
- Each model must use system data to fully quantify actual system coefficients
- Though most models are non-linear, we still tend to use linear tools such as least squares to find actual system coefficients
- To complicate this further, a large number of systems of interest tend to be unstable
- Instability makes collecting data very difficult – often we attempt a simple stabilization controller in order to collect data – how does this effect our ID process?
- Moreover, what kind of data does one need to determine a nonlinear model?



data

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2 \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -\frac{8}{3}x_3 + x_1x_2 \end{aligned}$$

model



prediction

# An example nonlinear system:

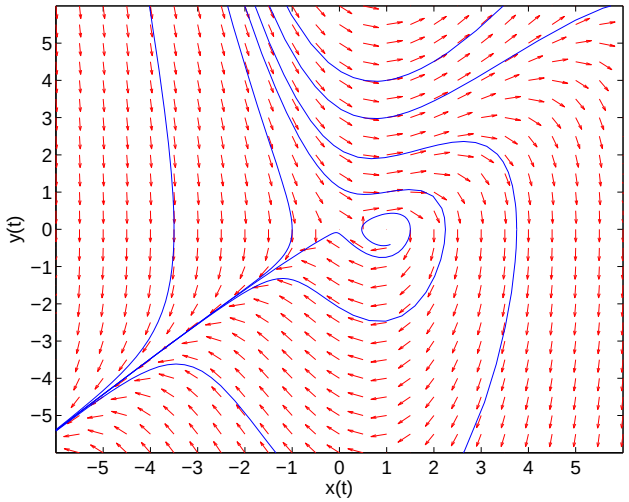
Consider performing SysID on a nonlinear constant coefficient homoclinic bifurcation system:

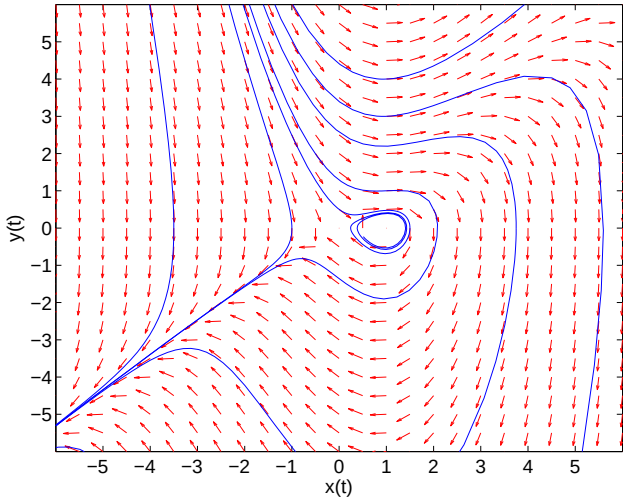
$$\begin{aligned}\dot{x}(t) &= y(t)\\ \dot{y}(t) &= \mu y(t) + x(t) + x^2(t) + x(t)y(t)\end{aligned}$$

top right phase portrait is for $\mu = -0.8$
lower right phase portrait is for $\mu = -0.95$

the system changes from the top unstable system with two fixed points to a trapped stable region around one of the fixed points near $\mu = -0.87$

both systems are generally unstable



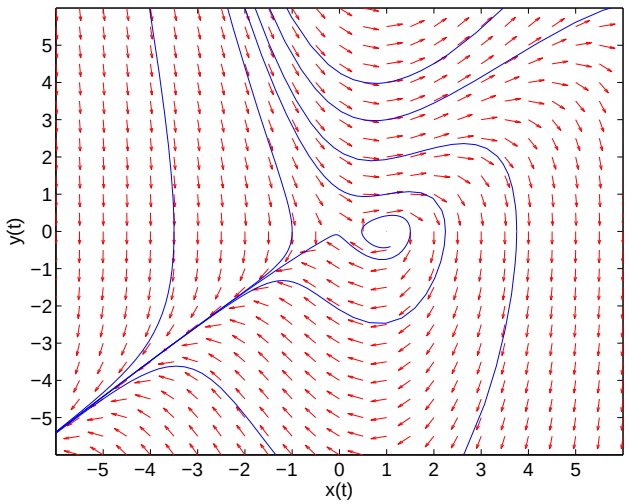



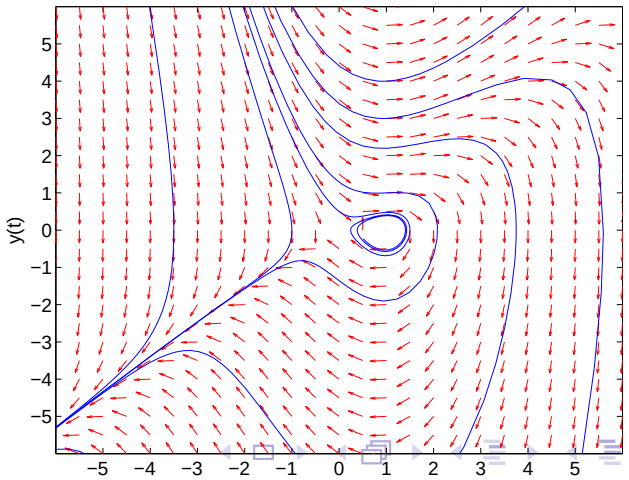
# What trajectories are needed to re-discover a system's dynamics?

- Trajectories in the upper left look a lot like a standard saddle manifold, does that matter?
- How many sample trajectories do we need?
- How long does each trajectory need to be?
- Do we need to sample around the fixed points?
- Do we need to sample in each quadrant?
- What happens when noise enters the picture?
- Do we need to know the exact equation?
- Can we stabilize the system so we can take more data – longer data?

These questions and more are summed up as:

**Persistent Excitation**



# In low noise conditions – say 100 db SNR:

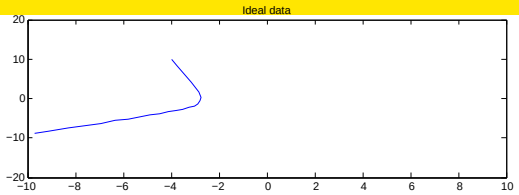
- Need only a single trajectory
- Any nonsingular trajectory will do
- Enough sample points to fully constrain the free coefficients
- This result is due to the uniqueness theorem of differential calculus
- The solution is a pseudo inverse:

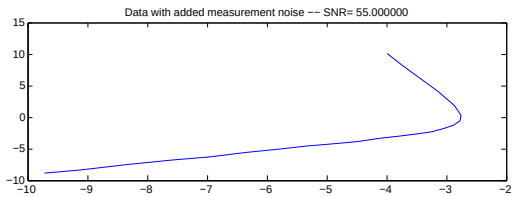
$$C = \left[ \left( \left(A^T A\right)^{-1} A^T \dot{x}(t) \right)^T \quad \left( \left(A^T A\right)^{-1} A^T \dot{y}(t) \right)^T \right]$$

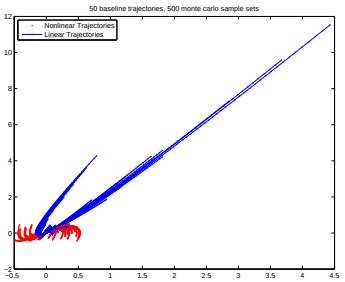
where $A = \left[ y \; x \; x^2 \; xy \right]$

**As noise grows this changes...**

**More data is required to overcome the added noise effects**




| C | $\dot{x}(t)$ coefficients | | | | $\dot{y}(t)$ coefficients | | | |
|---|---|---|---|---|---|---|---|---|
| SNR (dB) | $y$ | $x$ | $x^2$ | $xy$ | $y$ | $x$ | $x^2$ | $xy$ |
| 100.0 | +1.000 | -0.000 | -0.000 | +0.000 | -0.950 | +1.000 | -1.000 | +1.000 |
| 60.0 | +1.000 | -0.000 | -0.000 | +0.000 | -0.953 | +1.002 | -0.999 | +0.999 |
| 55.0 | +0.999 | +0.001 | +0.000 | -0.000 | -0.947 | +0.999 | -1.000 | +1.001 |

# Stabilizing the system:

One can make our example system fully controllable and observable by adding a control to the $\dot{x}(t)$:

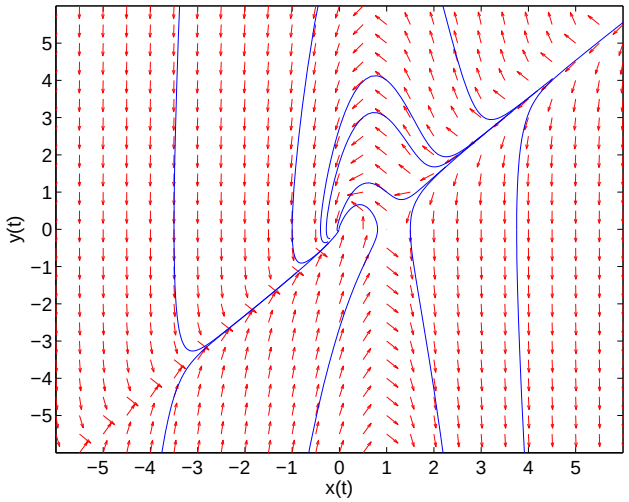
$$\begin{aligned}\dot{x}(t) &= y(t)+u(t)\\ \dot{y}(t) &= \mu y(t)+x(t)+x^2(t)+x(t)y(t)\end{aligned}$$

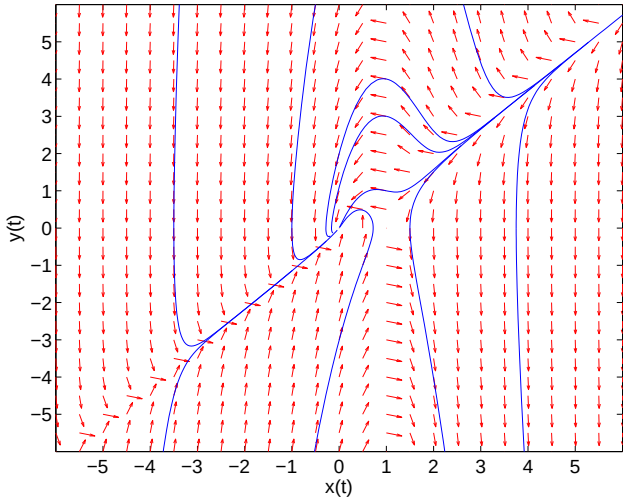
where $u(t)$ is the control. If one chooses the controller to be $u(t)=1.2(0-y(t))$, the system can be stabilized to $(0,0)$ for either system:

$$\begin{aligned}\dot{x}(t) &= -0.2y(t)\\ \dot{y}(t) &= \mu y(t)+x(t)+x^2(t)+x(t)y(t)\end{aligned}$$

the phase portraits transforms to:

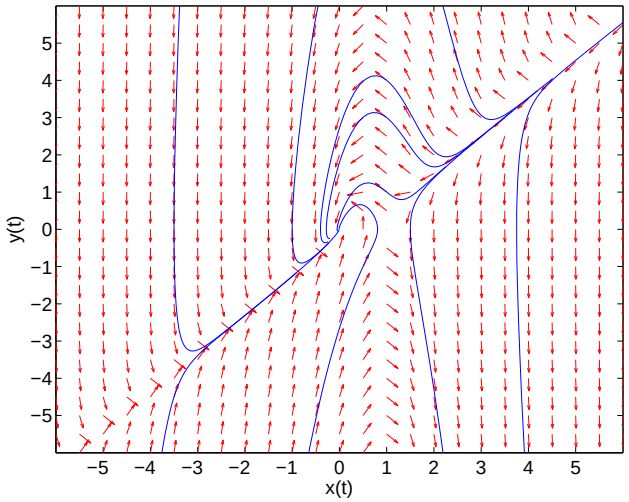
top right phase portrait is for $\mu=-0.8$
lower right phase portrait is for $\mu=-0.95$

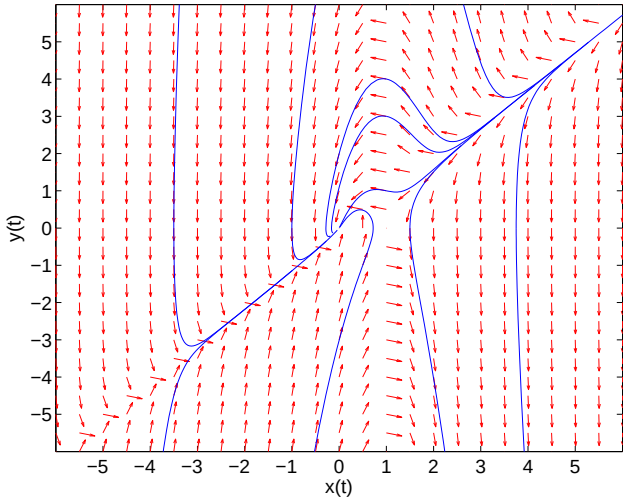


# What are the effects of stabilizing the system?

- Our basic linear modeling concept is – given a small enough region about a point – we can always approximate a nonlinear system with a local linear system
- The purpose of a regulator is to stabilize a system to a point
- Such a stabilization therefore drives the nonlinear effects out of our system data
- In fact don't the two systems to the right look even more similar now?
- Since our controller in essence drives the nonlinear effects out of the very system we are interested in learning about – why then control it?
- It is unstable and we need more data –
- How then has this helped us?
- How can we modify our approach and achieve our goals?



# Weighted least squares:

- Since control forces linearity out – weighted data based on its mismatch to a linear model:

$$\dot{x}(t) = -.2y(t)$$
$$\dot{y}(t) = -.95y(t) + x(t)$$

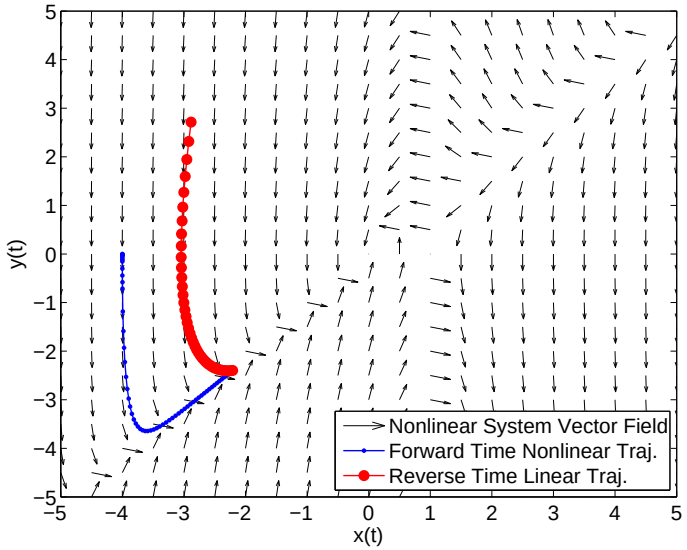
- How shall we weight?

Define $W[i]$ the weight for the $i^{th}$ sample point, $w[i]$:
$W[i] = ||w[i] - \hat{w}[i]|| \quad W = ||w - \hat{w}|| \forall i$

$\mathcal{W} = \max_i W[i]$

Normalizing results in: $W = \frac{W}{\mathcal{W}} + 1; W_{(i,j)} \in [1,2]$

- Now apply a weighted pseudo inverse:

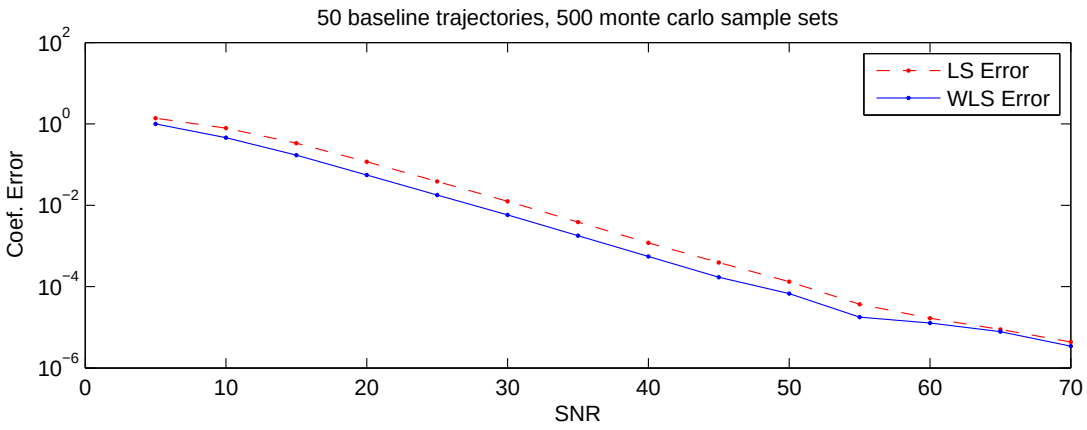


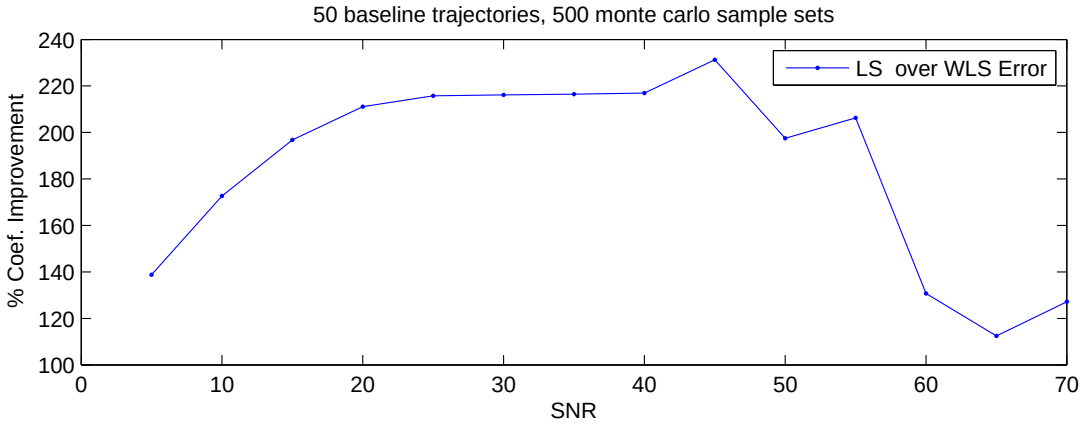

$$C = \left[ \left( \left(A^T W^T W A\right)^{-1} A^T W^T W \dot{x}(t) \right)^T \quad \left( \left(A^T W^T W A\right)^{-1} A^T W^T W \dot{y}(t) \right)^T \right] \text{ where } A = \left[ y \; x \; x^2 \; xy \right]$$

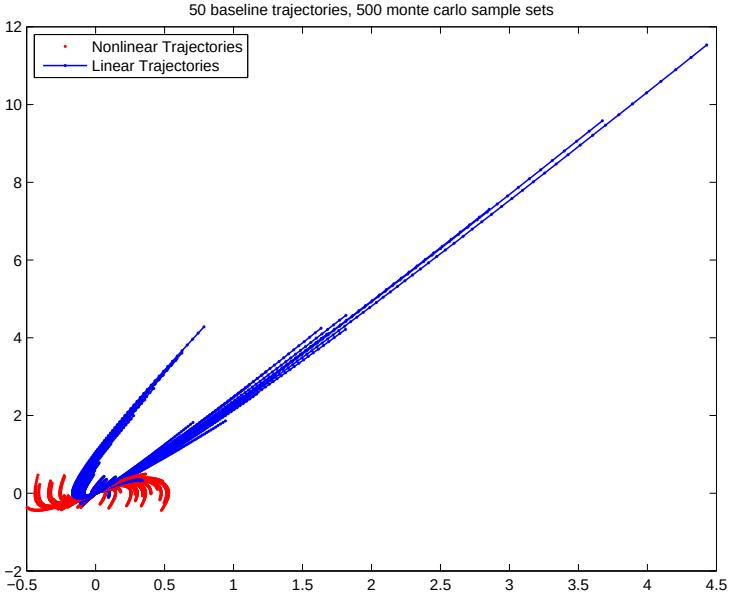
| | $\dot{x}(t)$ coefficients | | | | $\dot{y}(t)$ coefficients | | | |
|---|---|---|---|---|---|---|---|---|
| SNR (dB) | $y$ | $x$ | $x^2$ | $xy$ | $y$ | $x$ | $x^2$ | $xy$ |
| 100.0 | -0.200 | -0.000 | -0.000 | +0.000 | -0.950 | +1.000 | -1.000 | +1.000 |
| 85.0 | -0.200 | -0.000 | -0.000 | -0.000 | -0.951 | +1.001 | -1.000 | +1.000 |
| 70.0 | -0.200 | -0.000 | -0.000 | +0.000 | -0.942 | +0.992 | -1.002 | +1.002 |
| 55.0 | -0.201 | +0.001 | +0.000 | -0.000 | -0.939 | +0.985 | -1.003 | +1.002 |

# Results of weighting the nonlinear trajectory points:

WLS **improves** coefficient recovery by **5 db** over standard LS



# Conclusions and future directions

- This paper focuses on the discovery of nonlinear dynamics while under linear control
- One only needs the contributions of each mode to be larger than the noise
  - one can visit the region of phase-space dominated by that mode
  - weight the portions of the sampled dynamics that exhibit that model more heavily
- Our weighting function is only a starting place:
  - one might consider weighting across all signals instead of each individual signal
  - one might weight based on relative signal size and its length
  - one might consider balancing the amount of data caused by constant sample times and variations in dynamics evolution rates (e.g. trajectories close to fixed points evolve slower than trajectories farther away)
- Without some type of weighting scheme the amount of data contained within the slower evolving portions of the phase-space can simply just smooth out the effects of other portions of the phase-space