Resiliency of Linear System Consensus in the Presence of Channel Noise

Frank Ferrese[1]

Saroj Biswas[2]

Qing Dong[1]

Li Bai[2]

[1] Naval Surface Warfare Center, Carderock Division
Philadelphia, PA

[2] Temple University
Philadelphia, PA19122

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Outline

- Consensus Control Problem
- System Model
- Definition of consensus
- Stability Analysis
- Controller Resilience
  - Channel Failure
  - Node Failure
- Simulation results
- Conclusions

Objective

- Use multi-agent concepts for consensus control of networked large scale systems in the presence of channel noise.

  Assumptions:

  - Subsystems are linear time invariant
  - Subsystems are identical and cooperate for a common goal
  - Subsystems exchange sensor data with each other
  - Undirected data exchange through weighted interconnections
  - Control is based on weighted feedback of sensor data
  - Subsystems are perturbed by noise in the communication channel
  - Noise is Gaussian white

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System Model

Consider a networked control system of identical subsystems

$$\dot{x}_i = Ax_i + Bu_i, \quad i \in \{1, \cdots, N\}$$

$$y_i = Cx_i$$

$$u_i = K \sum_{j=1}^{N} a_{ij}(s_{ij} - y_i)$$

$$s_{ij} = y_j + \eta_{ij}$$

$x \in R^n$ is the state vector
$u \in R^m$ is the control input
$y \in R^r$ is the measurement data
s_{ij} is the sensor data received by node i from node j
η_{ij} are independent Gaussian white noise with mean zero
Covariance $E\{\eta_i(t_1)\eta_i^T(t_2)\} = Q\delta(t_1 - t_2)$
a_{ij} is the connection weight between subsystems i and j
K is a gain matrix
System matrices A, B, and C are constant.

Connectivity

Interconnection is undirected and connection weight $a_{ij} = a_{ji}$, and $a_{ij} = 0$ for no interconnection.

$$L = \left[\begin{array}{c|cccc} \sum_j a_{1j} & -a_{12} & -a_{13} & \cdots & -a_{1N} \\ \hline -a_{21} & \sum_j a_{2j} & -a_{23} & \cdots & -a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & -a_{N3} & \cdots & \sum_j a_{Nj} \end{array}\right] = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

For an undirected graph, the eigenvalues of L satisfy

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \leq \lambda^* = 2 \max_i \{d_i\} = 2 \max_i \sum_j a_{ij}$$

Connectivity

Let $\mathbf{1}_{N-1}$ be a $(N-1)$-dimensional column vector of 1's. Then for the transformation P

$$P = \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}$$

we have $$P^{-1}LP = \begin{bmatrix} 0 & L_{12} \\ \mathbf{0}_{N-1} & L_{22} - \mathbf{1}_{N-1}L_{12} \end{bmatrix}$$

Thus the eigenvalues of $L_{22} - \mathbf{1}_{N-1}L_{12}$ are same as the positive eigenvalues of the L matrix, i.e., $\{\lambda_2, \lambda_3, \cdots \lambda_N\}$.

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Definition

- *Mean Square Consensus:* The multiagent system is said to be have reached mean square consensus if

  $\lim_{t\to\infty} E\{\|x_i - x_j\|^2\} = 0 \quad \text{for any} \quad i \neq j$

  i.e., the subsystem trajectory will track that of the leader with zero mean square error.

- *Weak Mean Square Consensus:* The multiagent system is said to have reached weak mean square consensus if

  $\lim_{t\to\infty} E\{\|x_i - x_j\|^2\} = \varepsilon \quad \text{for any} \quad i \neq j$

  i.e., subsystem state will be only within ε-neighborhood of that of the leader.

- *Collective Weak Mean Square Consensus:* The multiagent system is said to have reached collective weak mean square consensus if

  $\lim_{t\to\infty} E\{\sum_{i=1}^{N} \|x_i - x_j\|^2\} = \varepsilon, \text{ for any } j$

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Main Result

Theorem. *Suppose the multiagent subsystem* $\{A, B, C\}$ *is controllable and observable, and* A *is stable. Denote* λ^* *as the the upper bound of the eigenvalues of the connectivity submatrix* $L_{22} - \mathbf{1}_{N-1} L_{12}$*. Then there exists a gain matrix* K *such that* $(A - \lambda^* BKC)$ *is negative definite, and the networked multiagent system is collectively weakly mean square consensusable, i.e., the leader-follower state error converges to the bound*

$$\lim_{t\to\infty} E \sum_{i\neq k} \|x_i(t) - x_k(t)\|^2 \to \frac{q}{\gamma_2}, \quad \textit{for any } k$$

where $q = \frac{1}{2}$*Trace*$\{H \otimes BKQK^T B^T\}$*,* H *is a diagonal matrix with* $h_{ii} = \sum_j a_{ij}^2 + \sum_j a_{1j}^2$*, and* $\gamma_2 = \gamma_2(K) > 0$*.*

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Proof

Take the first subsystem as the leader. Define the state error (disagreement) between subsystem i and the leader

$$\hat{x}_i = x_i - x_1, \quad 2 < i \leq N$$

Define

$$\hat{x} = [\hat{x}_2 \quad \hat{x}_3 \quad \ldots \quad \hat{x}_N]^T$$

Then the error dynamics can be described by

$$\dot{\hat{x}} = [(I_{N-1} \otimes A) - (L_{22} - \mathbf{1}_{N-1} L_{12}) \otimes BKC)]\hat{x} + (I_{N-1} \otimes BK)\,\psi$$

where the noise process $\psi_i(t) = \eta_i(t) - \eta_1(t)$.

Proof...

Rewrite the above equation as an Ito stochastic differential equation:

$$d\hat{x} = \hat{A}\hat{x}\,dt + \hat{B}\,dW$$

$$\text{where} \quad \hat{A} = [(I_{N-1} \otimes A) - (L_{22} - \mathbf{1}_{N-1}L_{12}) \otimes BKC)]$$

$$\hat{B} = (I_{N-1} \otimes BK)$$

$$W = \begin{bmatrix} W_1 & W_2 & \cdots & W_N \end{bmatrix}^T$$

where

$$E\{W(t)\} = 0$$

$$E\{W(t) - W(\tau))(W(t) - W(\tau))^T\} = (t - \tau)\hat{Q},$$

$$\hat{Q} = H \otimes Q$$

$$H = \text{diagonal},\ h_{ii} = \sum_j a_{ij}^2 + \sum_j a_{1j}^2,$$

where Q is the covariance matrix of the processes $\eta_i(t), i = 1, 2, \cdots N$, which are independent and have the same covariance matrix.

Proof...

Consider the process $\{V(t), t \geq 0\}$ defined by

$$V(t) = V(\hat{x}(t)) = \frac{1}{2}\|\hat{x}(t)\|^2$$

Using Ito's lemma

$$dV(t) = \left(\hat{x}^T \hat{A}\hat{x} + q\right) dt + \hat{x}^T \hat{B}\, dW$$

where

$$q = \frac{1}{2}\text{Trace}(\hat{B}\hat{Q}\hat{B}^T) = \frac{1}{2}\text{Trace}(H \otimes BKQK^T B^T)$$

Integrating over $(0, t)$ and taking expectation,

$$EV(t) = EV(0) + E\int_0^t \left(\hat{x}^T \hat{A}\hat{x}) + q\right) d\tau$$

Note that $\hat{A} = (I_{N-1} \otimes A) - (L_{22} - \mathbf{1}_{N-1} L_{12}) \otimes BKC$.

Proof...

Choose the gain matrix K so that $\hat{A}$ is negative definite with the bounds:

$$-\gamma_1 \|\hat{x}\|^2 \leq \hat{x}^T \hat{A} \hat{x} \leq -\gamma_2 \|\hat{x}\|^2, \qquad \gamma_1, \gamma_2 > 0$$

then

$$EV(t) \leq \frac{q}{2\gamma_2} + \left(EV(0) - \frac{q}{2\gamma_2}\right) e^{-2\gamma_2 t}$$

This shows that as $t \to \infty$,

$$EV(t) \to \frac{q}{2\gamma_2}$$

$$\lim_{t\to\infty} E\|\hat{x}(t)\|^2 \to \frac{q}{\gamma_2}$$

$$\lim_{t\to\infty} E \sum_i \|x_i(t) - x_1(t)\|^2 \to \frac{q}{\gamma_2}$$

The expected state error between the leader and the follower system converges to a certain limit.

Proof...

It can be shown[1] that the eigenvalues of $\hat{A} = [(I_{N-1} \otimes A) - (L_{22} - \mathbf{1}_{N-1} L_{12}) \otimes BKC)]$ are same as those of $A - \lambda_i BKC$, $i = 2, 3, \cdots N$,
where $\lambda_i > 0$ are the eigenvalues of L_{22}, with $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \leq \lambda^*$.

Assume the system $\{A, B, C\}$ is controllable and observable, and A is stable. Then there exists a $\hat{K} = \frac{K}{\lambda^*}$ so that for arbitrary $z \in R^n$

$$z^T(A - B\hat{K}C)z \leq -\delta_1 \|z\|^2, \quad \delta_1 > 0$$

$$z^T A z \leq -\delta_2 \|z\|^2, \quad \delta_2 \geq 0$$

$$\Rightarrow \quad z^T(A - \lambda_i BKC)z \leq -\Big(\frac{\lambda_i}{\lambda^*}\delta_1 + (1 - \frac{\lambda_i}{\lambda^*})\delta_2\Big)\|z\|^2 = -\gamma_2 \|z\|^2$$

This shows that $\hat{A}$ is negative definite for a suitably chosen K.

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[1] C.Q. Ma and J.F. Zhang, “Necessary and Sufficient Conditions for Consensusability of Linear Multi-Agent Systems", *IEEE Trans. on Automatic Control*, Vol. 55, 2010, pp. 1263-1268.

Controller Design

- The controller K is designed so that $\hat{A} = [(I_{N-1} \otimes A) - (L_{22} - \mathbf{1}_{N-1} L_{12}) \otimes BKC)]$ is stable, which in turn is equivalent to finding K so that $A - \lambda^* BKC$ is stable. For this purpose, one can use any of the known methods of feedback design of time invariant systems.
- The state error converges only to the limit $\lim_{t\to\infty} E\|\hat{x}(t)\|^2 \to \frac{q}{\gamma_2}$, where

$$q = \frac{1}{2}\text{Trace}(\hat{B}\hat{Q}\hat{B}^T) = \frac{1}{2}\text{Trace}(H \otimes BKQK^T B^T)$$
$$\hat{x}^T \hat{A}\hat{x} \leq -\gamma_2(K)\|\hat{x}\|^2,$$

  Note the gain matrix K in both the numerator and the denominator of the consensus error limit. This means that a larger value of K does not necessarily mean a smaller consensus error.
- A larger value of K implies larger γ_2, i.e., faster convergence, but at the same time a larger consensus error due to larger q
- There is a trade-off between rate of convergence and consensus error.

Remarks

- Requires the controllability and observability of the system $\{A, B, C\}$ to guarantee the existence of the gain matrix K.
- The system matrix A is assumed negative semidefinite. This is also necessary from a practical point of view since each subsystem is expected to operate as a standalone system or in the network in cooperation of other subsystems.
- In case the system evolves in a noise free environment, i.e., $q = 0$, the state error between any pair of subsystem to zero.
- In case the connection strength is same for any pair of subsystems, one can prove that the mean square state error between any pair of subsystems also converges to a small bound.

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Controller Resilience

The network connectivity matrix L_{22} is an integral component of this matrix.

Since the network connectivity can change because of a failure, the question is whether K would maintain stability.

Our proof only requires the upper bound of the eigenvalues of $L_{22} - \mathbf{1}_{N-1} L_{12}$. This leads to the invariance of the controller under different fault conditions. This is shown next.

Link Failure

Graph Laplacian for the network is given by:

$$L = \left[\begin{array}{c|cccc} \sum_j a_{1j} & -a_{12} & -a_{13} & \cdots & -a_{1N} \\ \hline -a_{21} & \sum_j a_{2j} & -a_{23} & \cdots & -a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & -a_{N3} & \cdots & \sum_j a_{Nj} \end{array}\right]$$

If the communication link between nodes m and n fails, we set $a_{mn} = a_{nm} = 0$.
Thus $d_n = \sum_j a_{nj}$ will be smaller than their corresponding pre-fault values,
however all eigenvalues of L of faulted system remain bounded by the same pre-fault λ^*.

$$0 = \lambda_1^f < \lambda_2^f \leq \lambda_3^f \leq \cdots \leq \lambda_N^f \leq 2 \max_i \{a_{ij}^f\} \leq \lambda^* = 2 \max_i \sum_j a_{ij}$$

$$z^T(A - \lambda_i^f BKC)z \leq -\left(\tfrac{\lambda_i}{\lambda^*}\delta_1 + (1 - \tfrac{\lambda_i^f}{\lambda^*})\delta_2\right)\|z\|^2 < 0 \quad \text{for all } i = 2, 3, \cdots, N$$

Thus the pre-fault controller remains a valid controller for the post-fault system.

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Node Failure

Suppose the subsystem k fails. Then we set $a_{ik} = a_{ki} = 0$ for all i.

$$L_N^{\mathrm{f}} = \begin{bmatrix} d_1^{\mathrm{f}} & -a_{12} & -a_{13} & \cdots & 0 & \cdots & -a_{1N} \\ -a_{21} & d_2^{\mathrm{f}} & -a_{23} & \cdots & 0 & \cdots & -a_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{N1} & -a_{N2} & -a_{N3} & \cdots & 0 & \cdots & d_N^{\mathrm{f}} \end{bmatrix}$$

The degree d_i^{f} of node i of the failed system will be bounded by that of the pre-fault system

$$d_i^{\mathrm{f}} = \sum_{j \neq k} a_{ij} \leq \sum_j a_{ij} = d_i^{\mathrm{o}}$$

$$\text{and} \quad \{\lambda_1^{\mathrm{f}} = 0, \lambda_2^{\mathrm{f}} = 0 < \lambda_3^{\mathrm{f}} \leq \lambda_4^{\mathrm{f}} \leq \cdots \leq \lambda_N^{\mathrm{f}}\} \leq \lambda^*$$

Note that there are now two zeros in the set of eigenvalues.

The pre-fault controller maintains stability of the post-fault system.

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Example

$$\begin{bmatrix} \dot{x}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$$

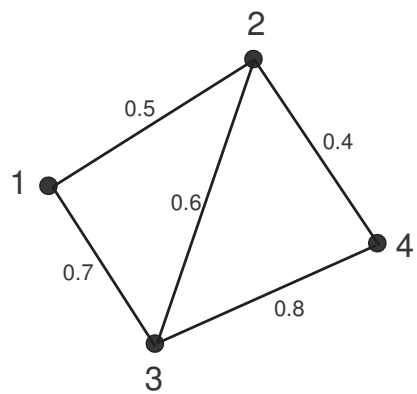
$$y_i = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

$$u_i = K \sum_{j=1}^{N} a_{ij}(s_{ij} - y_i)$$

$$s_{ij} = y_j + \eta_{ij}$$

where η_{ij} is a Gaussian white noise with mean zero and covariance $\sigma^2 = 0.1$.

Network Topology

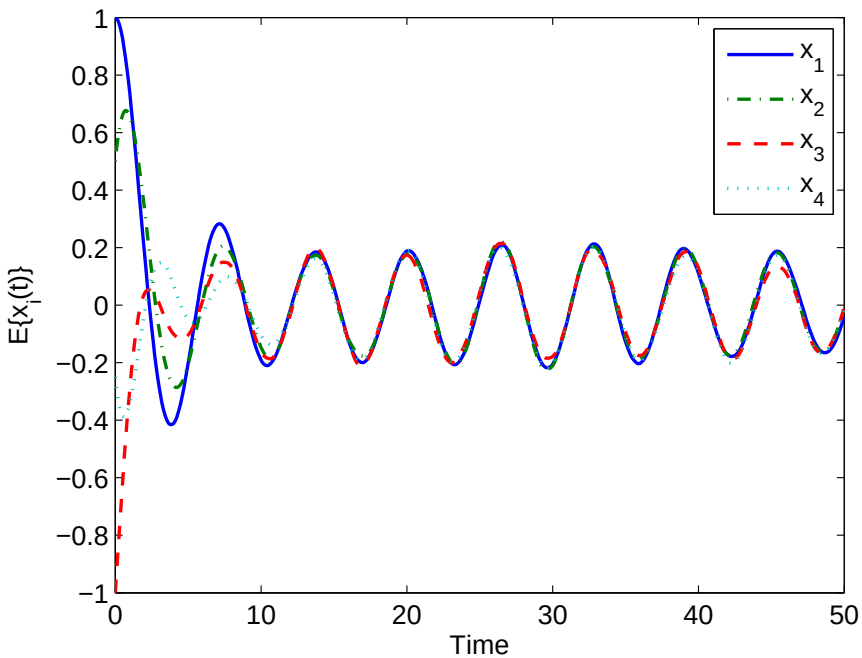


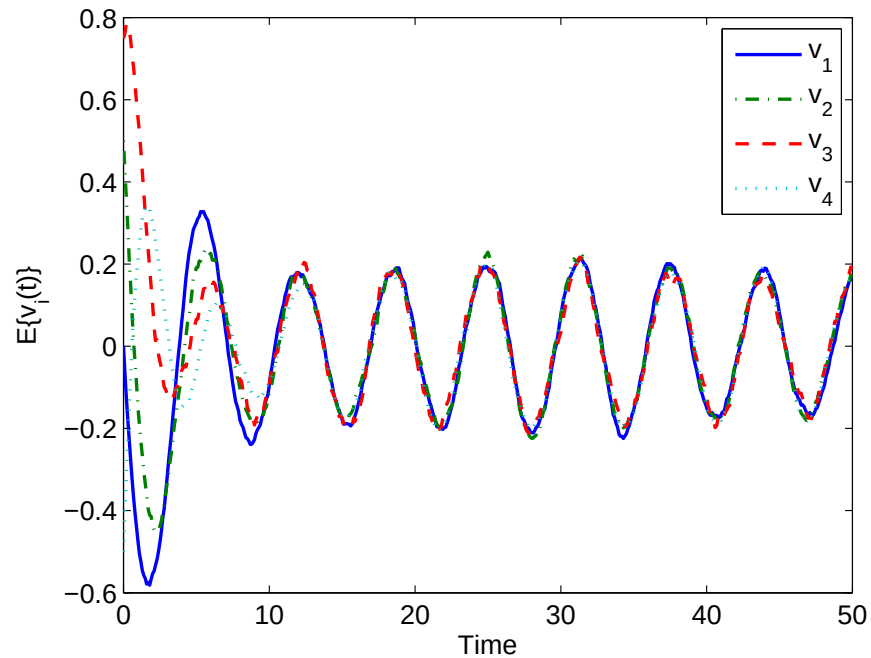
$$L = \begin{bmatrix} 1.2 & -0.5 & -0.7 & 0 \\ -0.5 & 1.5 & -0.6 & -0.4 \\ -0.7 & -0.6 & 2.1 & -0.8 \\ 0 & -0.4 & -0.8 & 1.2 \end{bmatrix}$$

which has the eigenvalues $\lambda = \{0, 1.1906, 1.9821, 2.8273\}$, and $\lambda^* = 4.2$.

For the gain, we choose $K = 0.5$.

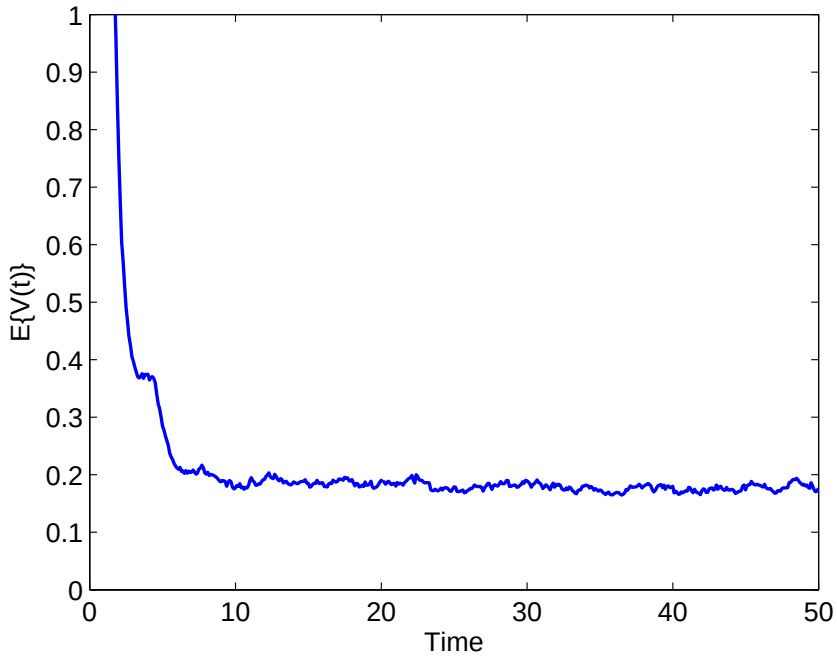
Example – Consensus





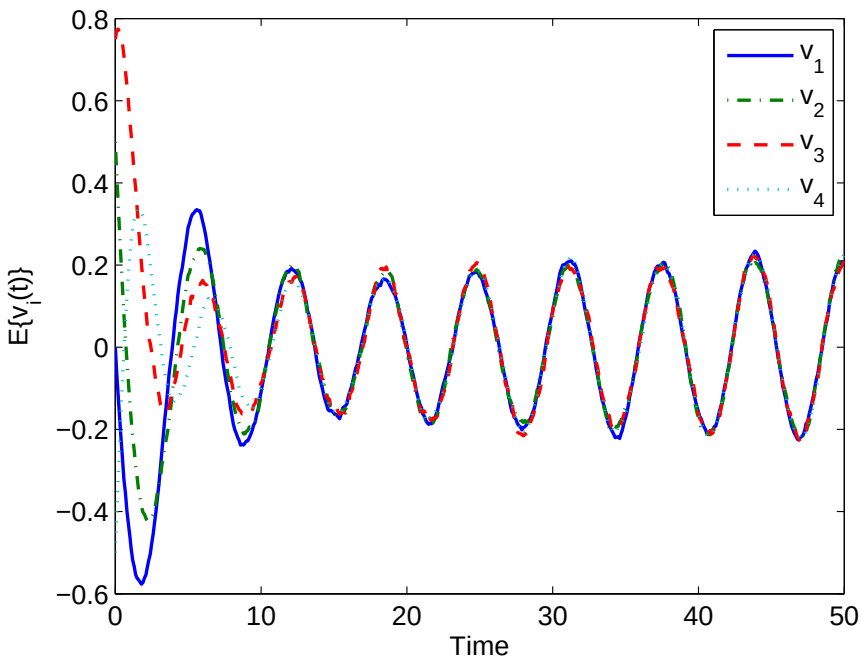
Ensemble average of system state x_i and v_i

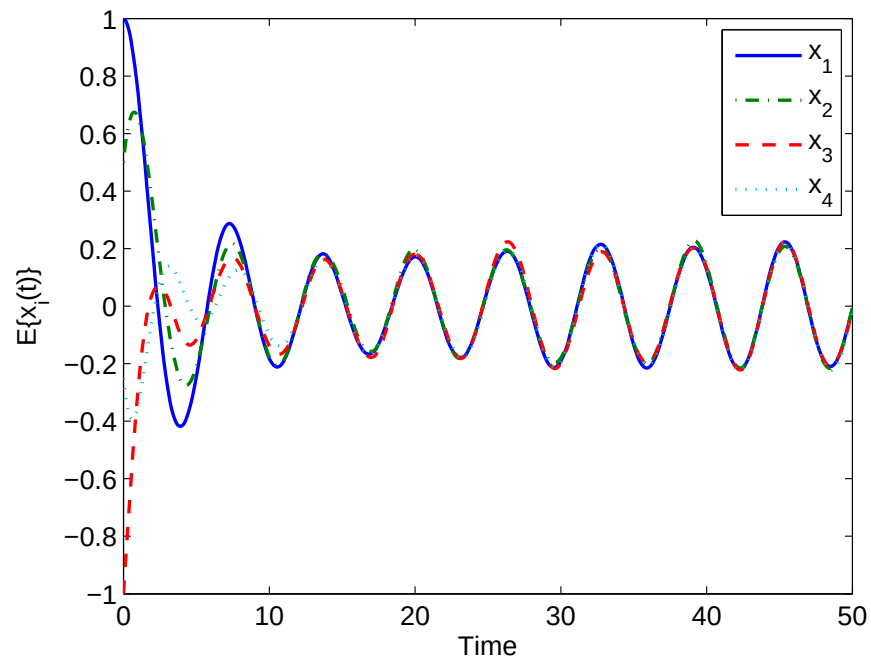
Example – Consensus



Ensemble average of energy $V(t)$

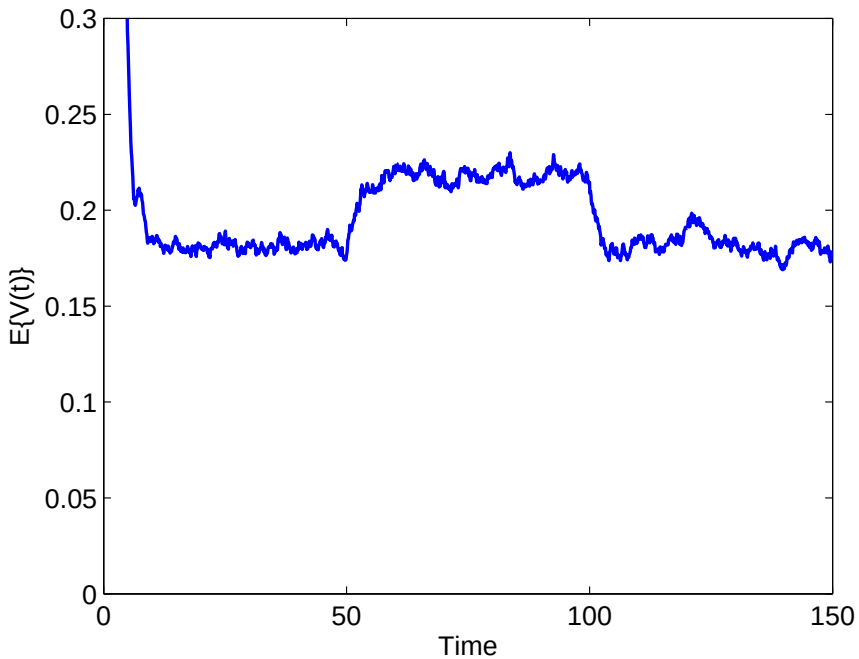
Link Failure





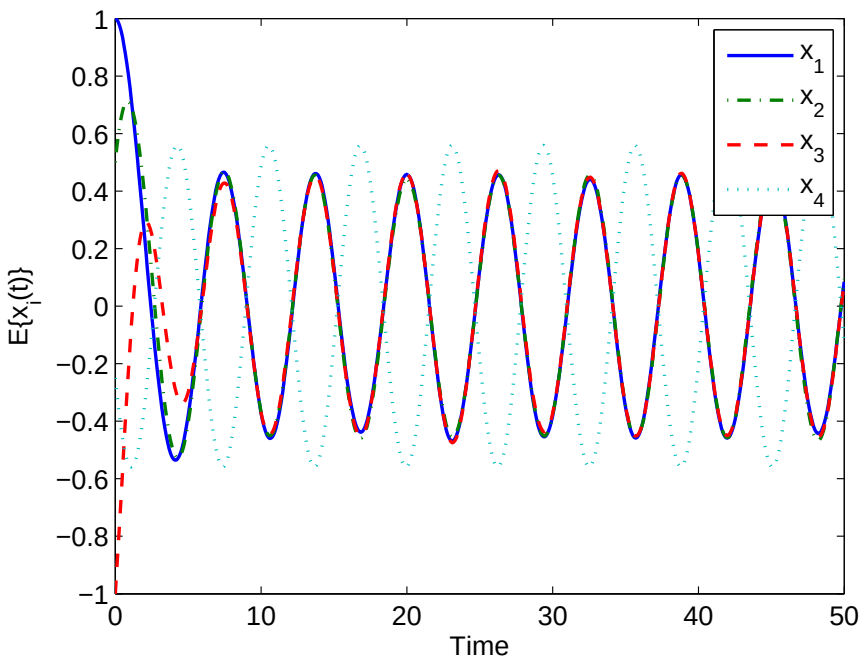
Ensemble average of system state x_i and v_i with link (2,3) failed

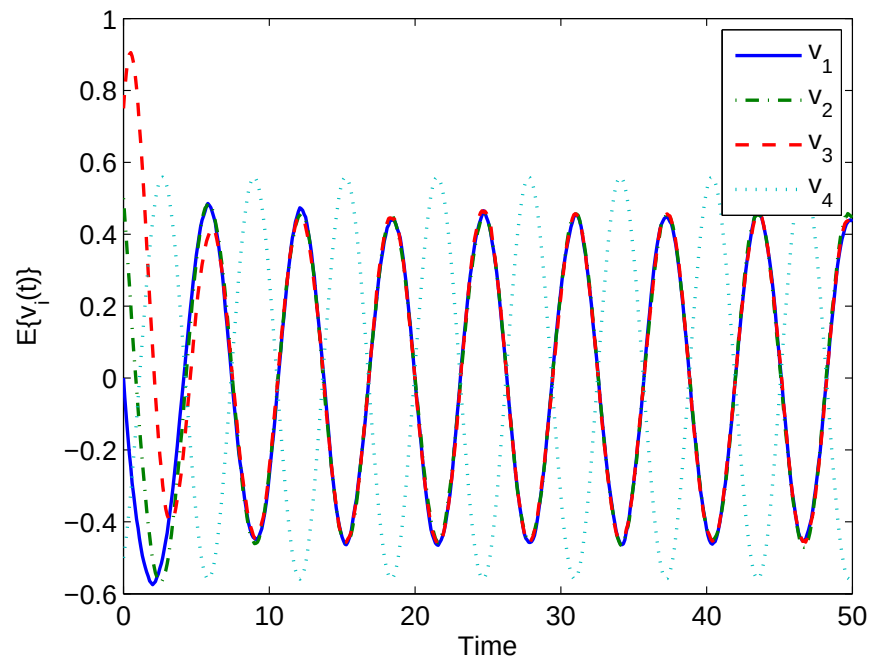
Example – Link Failure



Link (2,3) failed at $t = 50$ and recovered at $t = 100$. A link failure has two contradicting effects: 1) Less noise enters the system – a stabilizing effect, and 2) Loss of sensor signal for control – a destabilizing effect. The net result is a loss of system performance in the sense of a larger consensus error.

Node Failure

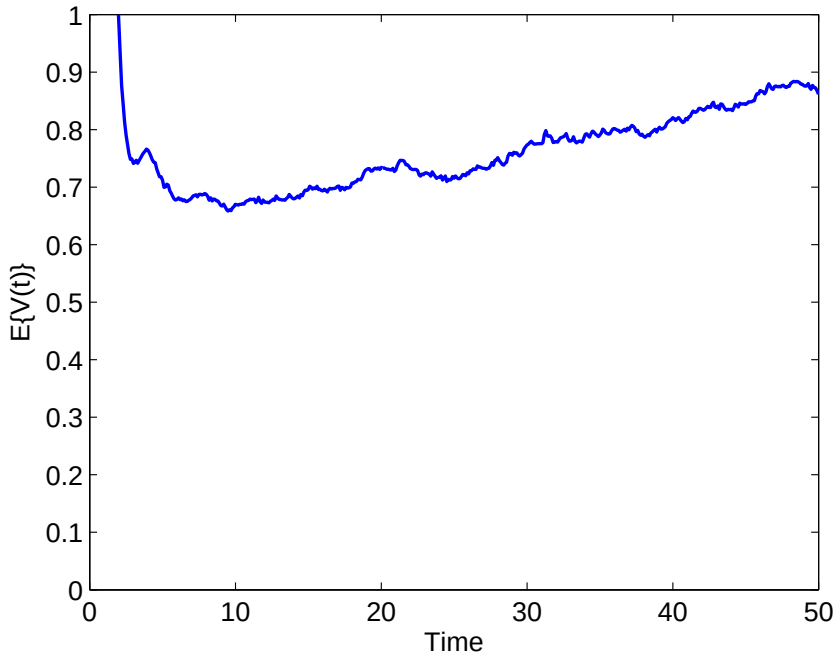




Ensemble average of system state x_i and v_i with Node 4 failed

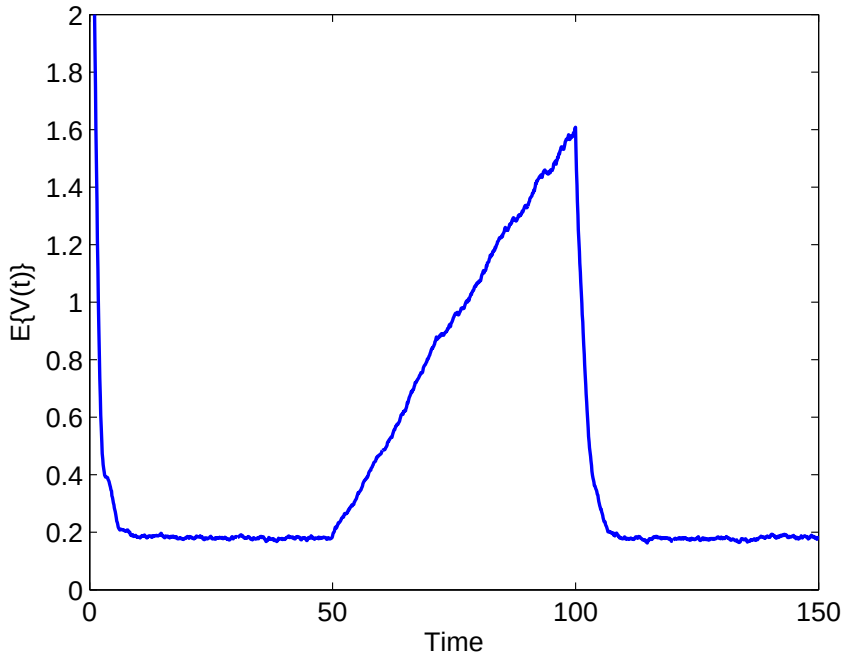
Here node failure is assumed to be failure of its control system and loss of all sensor signal for other subsystems.

Node Failure



Ensemble average of energy $V(t)$ with node 4 failed

Node Failure



Node 4 failed at $t = 50$ and recovered at $t = 100$. Since the system without a controller is oscillatory, it remained in that state leading to a large consensus error. Normal performance was resumed after the controller was added.

Conclusions

- Multiagent concept has been used for consensus control of linear systems.
- The overall system consists of an interconnection of many identical subsystems.
- Connectivity between subsystems is undirected with nonuniform interconnection weights.
- The overall system evolves in a noisy environment with channel noise.
- Collective consensus is achieved in the weak mean square sense, i.e., subsystems collectively track the leader state only within a small bound. This is expected since the system is noisy.
- The controller is resilient to communication or subsystem failures in the sense that it maintains collective stability of the interconnected system.

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