Resiliency of Linear System Consensus in the Presence of Channel Noise

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August, 2012
Outline

- Consensus Control Problem
- System Model
- Definition of consensus
- Stability Analysis
- Controller Resilience
  - Channel Failure
  - Node Failure
- Simulation results
- Conclusions
Objective

Use multi-agent concepts for consensus control of networked large scale systems in the presence of channel noise.

Assumptions:

- Subsystems are linear time invariant
- Subsystems are identical and cooperate for a common goal
- Subsystems exchange sensor data with each other
- Undirected data exchange through weighted interconnections
- Control is based on weighted feedback of sensor data
- Subsystems are perturbed by noise in the communication channel
- Noise is Gaussian white
System Model

Consider a networked control system of identical subsystems

\[ \dot{x}_i = Ax_i + Bu_i, \quad i \in \{1, \cdots, N\} \]
\[ y_i = Cx_i \]
\[ u_i = K \sum_{j=1}^{N} a_{ij} (s_{ij} - y_i) \]
\[ s_{ij} = y_j + \eta_{ij} \]

- \( x \in \mathbb{R}^n \) is the state vector
- \( u \in \mathbb{R}^m \) is the control input
- \( y \in \mathbb{R}^r \) is the measurement data
- \( s_{ij} \) is the sensor data received by node \( i \) from node \( j \)
- \( \eta_{ij} \) are independent Gaussian white noise with mean zero
- Covariance \( E\{\eta_i(t_1)\eta_i^T(t_2)\} = Q \delta(t_1 - t_2) \)
- \( a_{ij} \) is the connection weight between subsystems \( i \) and \( j \)
- \( K \) is a gain matrix
- System matrices \( A, B, \) and \( C \) are constant.
Connectivity

Interconnection is undirected and connection weight $a_{ij} = a_{ji}$, and $a_{ij} = 0$ for no interconnection.

$$L = \begin{bmatrix}
\sum_j a_{1j} & -a_{12} & -a_{13} & \cdots & -a_{1N} \\
-a_{21} & \sum_j a_{2j} & -a_{23} & \cdots & -a_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{N1} & -a_{N2} & -a_{N3} & \cdots & \sum_j a_{Nj}
\end{bmatrix} = \begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}$$

For an undirected graph, the eigenvalues of $L$ satisfy

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \leq \lambda^* = 2 \max_i \{d_i\} = 2 \max_i \sum_j a_{ij}$$
Connectivity

Let $\mathbf{1}_{N-1}$ be a $(N - 1)$-dimensional column vector of 1’s. Then for the transformation $P$

$$P = \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{bmatrix}$$

we have

$$P^{-1}LP = \begin{bmatrix} 0 & L_{12} \\ 0_{N-1} & L_{22} - \mathbf{1}_{N-1}L_{12} \end{bmatrix}$$

Thus the eigenvalues of $L_{22} - \mathbf{1}_{N-1}L_{12}$ are same as the positive eigenvalues of the $L$ matrix, i.e., $\{\lambda_2, \lambda_3, \cdots \lambda_N\}$. 
Definition

- **Mean Square Consensus:** The multiagent system is said to have reached mean square consensus if
  \[
  \lim_{t \to \infty} E\{\|x_i - x_j\|^2\} = 0 \quad \text{for any} \quad i \neq j
  \]
i.e., the subsystem trajectory will track that of the leader with zero mean square error.

- **Weak Mean Square Consensus:** The multiagent system is said to have reached weak mean square consensus if
  \[
  \lim_{t \to \infty} E\{\|x_i - x_j\|^2\} = \varepsilon \quad \text{for any} \quad i \neq j
  \]
i.e., subsystem state will be only within \(\varepsilon\)-neighborhood of that of the leader.

- **Collective Weak Mean Square Consensus:** The multiagent system is said to have reached collective weak mean square consensus if
  \[
  \lim_{t \to \infty} E\{\sum_{i=1}^{N} \|x_i - x_j\|^2\} = \varepsilon, \quad \text{for any} \quad j
  \]
Main Result

**Theorem.** Suppose the multiagent subsystem \( \{A, B, C\} \) is controllable and observable, and \( A \) is stable. Denote \( \lambda^* \) as the upper bound of the eigenvalues of the connectivity submatrix \( L_{22} - 1_{N-1} L_{12} \). Then there exists a gain matrix \( K \) such that \( (A - \lambda^* BKC) \) is negative definite, and the networked multiagent system is collectively weakly mean square consensusable, i.e., the leader-follower state error converges to the bound

\[
\lim_{t \to \infty} E \sum_{i \neq k} \|x_i(t) - x_k(t)\|^2 \to \frac{q}{\gamma^2}, \quad \text{for any } k
\]

where \( q = \frac{1}{2} \text{Trace}\{H \otimes B K Q K^T B^T\} \), \( H \) is a diagonal matrix with \( h_{ii} = \sum_j a_{ij}^2 + \sum_j a_{1j}^2 \), and \( \gamma^2 = \gamma^2(K) > 0 \).
Proof

Take the first subsystem as the leader. Define the state error (disagreement) between subsystem $i$ and the leader

$$\hat{x}_i = x_i - x_1, \quad 2 < i \leq N$$

Define

$$\hat{x} = [\hat{x}_2 \quad \hat{x}_3 \ldots \quad \hat{x}_N]^T$$

Then the error dynamics can be described by

$$\dot{\hat{x}} = [(I_{N-1} \otimes A) - (L_{22} - 1_{N-1}L_{12}) \otimes BKC)]\hat{x} + (I_{N-1} \otimes BK)\psi$$

where the noise process $\psi_i(t) = \eta_i(t) - \eta_1(t)$. 
Proof...

Rewrite the above equation as an Ito stochastic differential equation:

\[ d\hat{x} = \hat{A}\hat{x} \, dt + \hat{B} \, dW \]

where \( \hat{A} = [(I_{N-1} \otimes A) - (L_{22} - 1_{N-1}L_{12}) \otimes BK C)] \)

\[ \hat{B} = (I_{N-1} \otimes BK) \]

\[ W = \begin{bmatrix} W_1 & W_2 & \cdots & W_N \end{bmatrix}^T \]

where

\[ E\{W(t)\} = 0 \]

\[ E\{W(t) - W(\tau))(W(t) - W(\tau))^T\} = (t - \tau)\hat{Q}, \]

\[ \hat{Q} = H \otimes Q \]

\[ H = \text{diagonal}, \quad h_{ii} = \sum_j a_{i,j}^2 + \sum_j a_{1,j}^2, \]

where \( Q \) is the covariance matrix of the processes \( \eta_i(t), i = 1, 2, \cdots N \), which are independent and have the same covariance matrix.
Consider the process \( \{V(t), t \geq 0\} \) defined by

\[
V(t) = V(\hat{x}(t)) = \frac{1}{2} \|\hat{x}(t)\|^2
\]

Using Ito’s lemma

\[
dV(t) = (\hat{x}^T \hat{A} \hat{x} + q) \, dt + \hat{x}^T \hat{B} \, dW
\]

where

\[
q = \frac{1}{2} \text{Trace}(\hat{B} \hat{Q} \hat{B}^T) = \frac{1}{2} \text{Trace}(H \otimes BKQK^T BT)
\]

Integrating over \((0, t)\) and taking expectation,

\[
EV(t) = EV(0) + E \int_0^t (\hat{x}^T \hat{A} \hat{x}) + q \, d\tau
\]

Note that \( \hat{A} = (I_{N-1} \otimes A) - (L_{22} - 1_{N-1} L_{12}) \otimes BK C. \)
Proof...

Choose the gain matrix $K$ so that $\hat{A}$ is negative definite with the bounds:

$$-\gamma_1 \|\hat{x}\|^2 \leq \hat{x}^T \hat{A} \hat{x} \leq -\gamma_2 \|\hat{x}\|^2, \quad \gamma_1, \gamma_2 > 0$$

then

$$EV(t) \leq \frac{q}{2\gamma_2} + (EV(0) - \frac{q}{2\gamma_2}) e^{-2\gamma_2 t}$$

This shows that as $t \to \infty$,

$$EV(t) \to \frac{q}{2\gamma_2}$$

$$\lim_{t \to \infty} E\|\hat{x}(t)\|^2 \to \frac{q}{\gamma_2}$$

$$\lim_{t \to \infty} E\sum_{i} \|x_i(t) - x_1(t)\|^2 \to \frac{q}{\gamma_2}$$

The expected state error between the leader and the follower system converges to a certain limit.
Proof...

It can be shown\(^1\) that the eigenvalues of \(\hat{A} = [(I_{N-1} \otimes A) - (L_{22} - 1_{N-1}L_{12}) \otimes BKC)]\) are same as those of \(A - \lambda_i BKC, \ i = 2, 3, \cdots N\), where \(\lambda_i > 0\) are the eigenvalues of \(L_{22}\), with \(0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N \leq \lambda^*\).

Assume the system \(\{A, B, C\}\) is controllable and observable, and \(A\) is stable. Then there exists a \(\hat{K} = K_0\) so that for arbitrary \(z \in R^n\)

\[
z^T (A - B\hat{K} C) z \leq -\delta_1 \|z\|^2, \quad \delta_1 > 0
\]

\[
z^T A z \leq -\delta_2 \|z\|^2, \quad \delta_2 \geq 0
\]

\[
\Rightarrow \quad z^T (A - \lambda_i BKC) z \leq -\left(\frac{\lambda_i}{\lambda^*} \delta_1 + (1 - \frac{\lambda_i}{\lambda^*})\delta_2\right) \|z\|^2 = -\gamma_2 \|z\|^2
\]

This shows that \(\hat{A}\) is negative definite for a suitably chosen \(K\).

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Controller Design

- The controller $K$ is designed so that
  \[ \hat{A} = [(I_{N-1} \otimes A) - (L_{22} - 1_{N-1} L_{12}) \otimes BKC)] \]
  is stable, which in turn is equivalent to finding $K$ so that $A - \lambda^* BKC$ is stable. For this purpose, one can use any of the known methods of feedback design of time invariant systems.

- The state error converges only to the limit $\lim_{t \to \infty} E\|\hat{x}(t)\|^2 \to \frac{q}{\gamma_2}$, where
  \[ q = \frac{1}{2} \text{Trace}(\hat{BQ}\hat{B}^T) = \frac{1}{2} \text{Trace}(H \otimes BKQK^TB^T) \]
  \[ \hat{x}^T \hat{A} \hat{x} \leq -\gamma_2(K)\|\hat{x}\|^2, \]

  Note the gain matrix $K$ in both the numerator and the denominator of the consensus error limit. This means that a larger value of $K$ does not necessarily mean a smaller consensus error.

- A larger value of $K$ implies larger $\gamma_2$, i.e., faster convergence, but at the same time a larger consensus error due to larger $q$.

- There is a trade-off between rate of convergence and consensus error.
Remarks

- Requires the controllability and observability of the system \( \{A, B, C\} \) to guarantee the existence of the gain matrix \( K \).

- The system matrix \( A \) is assumed negative semidefinite. This is also necessary from a practical point of view since each subsystem is expected to operate as a standalone system or in the network in cooperation of other subsystems.

- In case the system evolves in a noise free environment, i.e., \( q = 0 \), the state error between any pair of subsystem to zero.

- In case the connection strength is same for any pair of subsystems, one can prove that the mean square state error between any pair of subsystems also converges to a small bound.
Controller Resilience

The network connectivity matrix $L_{22}$ is an integral component of this matrix.

Since the network connectivity can change because of a failure, the question is whether $K$ would maintain stability.

Our proof only requires the upper bound of the eigenvalues of $L_{22} - 1_{N-1}L_{12}$. This leads to the invariance of the controller under different fault conditions. This is shown next.
Graph Laplacian for the network is given by:

\[ L = \begin{bmatrix}
\sum_j a_{1j} & -a_{12} & -a_{13} & \cdots & -a_{1N} \\
-a_{21} & \sum_j a_{2j} & -a_{23} & \cdots & -a_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{N1} & -a_{N2} & -a_{N3} & \cdots & \sum_j a_{Nj}
\end{bmatrix} \]

If the communication link between nodes \( m \) and \( n \) fails, we set \( a_{mn} = a_{nm} = 0 \).

Thus \( d_n = \sum_j a_{nj} \) will be smaller than their corresponding pre-fault values, however all eigenvalues of \( L \) of faulted system remain bounded by the same pre-fault \( \lambda^* \).

\[ 0 = \lambda^*_1 < \lambda^*_2 \leq \lambda^*_3 \leq \cdots \leq \lambda^*_N \leq 2 \max_i \{a^f_{ij}\} \leq \lambda^* = 2 \max_i \sum_j a_{ij} \]

\[ z^T (A - \lambda^f_i BKC) z \leq -\left(\frac{\lambda^*_i}{\lambda^*} \delta_1 + (1 - \frac{\lambda^*_i}{\lambda^*}) \delta_2\right) \|z\|^2 < 0 \quad \text{for all } i = 2, 3, \cdots, N \]

Thus the pre-fault controller remains a valid controller for the post-fault system.
Node Failure

Suppose the subsystem $k$ fails. Then we set $a_{ik} = a_{ki} = 0$ for all $i$.

$$L_N^f = \begin{bmatrix}
  d_1^f & -a_{12} & -a_{13} & \cdots & 0 & \cdots & -a_{1N} \\
  -a_{21} & d_2^f & -a_{23} & \cdots & 0 & \cdots & -a_{2N} \\
  0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  -a_{N1} & -a_{N2} & -a_{N3} & \cdots & 0 & \cdots & d_N^f
\end{bmatrix}$$

The degree $d_i^f$ of node $i$ of the failed system will be bounded by that of the pre-fault system

$$d_i^f = \sum_{j \neq k} a_{ij} \leq \sum_j a_{ij} = d_i^o$$

and \( \{ \lambda_1^f = 0, \lambda_2^f = 0 < \lambda_3^f \leq \lambda_4^f \leq \cdots \leq \lambda_N^f \} \leq \lambda^* \)

Note that there are now two zeros in the set of eigenvalues. The pre-fault controller maintains stability of the post-fault system.
Example

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{v}_i
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_i \\
v_i
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u_i
\]

\[
y_i = \begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_i \\
v_i
\end{bmatrix}
\]

\[
u_i = K \sum_{j=1}^{N} a_{ij} (s_{ij} - y_i)
\]

\[
s_{ij} = y_j + \eta_{ij}
\]

where \(\eta_{ij}\) is a Gaussian white noise with mean zero and covariance \(\sigma^2 = 0.1\).
Network Topology

\[ L = \begin{bmatrix}
1.2 & -0.5 & -0.7 & 0 \\
-0.5 & 1.5 & -0.6 & -0.4 \\
-0.7 & -0.6 & 2.1 & -0.8 \\
0 & -0.4 & -0.8 & 1.2
\end{bmatrix} \]

which has the eigenvalues \( \lambda = \{0, 1.1906, 1.9821, 2.8273\} \), and \( \lambda^* = 4.2 \).

For the gain, we choose \( K = 0.5 \).
Example – Consensus

Ensemble average of system state $x_i$ and $v_i$
Example – Consensus

Ensemble average of energy $V(t)$
Link Failure

Ensemble average of system state $x_i$ and $v_i$ with link (2,3) failed
Example – Link Failure

Link (2,3) failed at $t = 50$ and recovered at $t = 100$. A link failure has two contradicting effects: 1) Less noise enters the system – a stabilizing effect, and 2) Loss of sensor signal for control – a destabilizing effect. The net result is a loss of system performance in the sense of a larger consensus error.
Node Failure

Ensemble average of system state $x_i$ and $v_i$ with Node 4 failed

Here node failure is assumed to be failure of its control system and loss of all sensor signal for other subsystems.
Node Failure

Ensemble average of energy $V(t)$ with node 4 failed
Node 4 failed at $t = 50$ and recovered at $t = 100$. Since the system without a controller is oscillatory, it remained in that state leading to a large consensus error. Normal performance was resumed after the controller was added.
Conclusions

- Multiagent concept has been used for consensus control of linear systems.
- The overall system consists of an interconnection of many identical subsystems.
- Connectivity between subsystems is undirected with nonuniform interconnection weights.
- The overall system evolves in a noisy environment with channel noise.
- Collective consensus is achieved in the weak mean square sense, i.e., subsystems collectively track the leader state only within a small bound. This is expected since the system is noisy.
- The controller is resilient to communication or subsystem failures in the sense that it maintains collective stability of the interconnected system.
Acknowledgement

The authors are thankful to the Office of Naval Research, especially Anthony Seman for his support on this research.
Thank you!